

GEOMETRIC COLLECTIONS AND CASTELNUOVO-MUMFORD REGULARITY

L. COSTA*, R.M. MIRÓ-ROIG**

ABSTRACT. The paper begins by overviewing the basic facts on geometric exceptional collections. Then, we derive, for any coherent sheaf \mathcal{F} on a smooth projective variety with a geometric collection, two spectral sequences: the first one abuts to \mathcal{F} and the second one to its cohomology. The main goal of the paper is to generalize Castelnuovo-Mumford regularity for coherent sheaves on projective spaces to coherent sheaves on smooth projective varieties X with a geometric collection σ . We define the notion of regularity of a coherent sheaf \mathcal{F} on X with respect to σ . We show that the basic formal properties of the Castelnuovo-Mumford regularity of coherent sheaves over projective spaces continue to hold in this new setting and we show that in case of coherent sheaves on \mathbb{P}^n and for a suitable geometric collection of coherent sheaves on \mathbb{P}^n both notions of regularity coincide. Finally, we carefully study the regularity of coherent sheaves on a smooth quadric hypersurface $Q_n \subset \mathbb{P}^{n+1}$ (n odd) with respect to a suitable geometric collection and we compare it with the Castelnuovo-Mumford regularity of their extension by zero in \mathbb{P}^{n+1} .

CONTENTS

1. Introduction	1
2. Geometric collections and spectral sequences	3
3. Regularity with respect to geometric collections: definition and properties	11
4. Regularity of sheaves on Q_n	16
5. Final comments	23
References	24

1. INTRODUCTION

The goal of this paper is to extend the notion of Castelnuovo-Mumford regularity for coherent sheaves on projective spaces to coherent sheaves on smooth projective varieties with a geometric collection with the hope to apply it to study coherent sheaves on smooth projective varieties. As a main tool we use geometric collections of exceptional sheaves and helix theory.

Exceptional bundles were first considered in [7] by Drezet and Le Potier, where they were used to determine the set of triples (r, c_1, c_2) such that there exists a semistable sheaf

Date: February 2, 2008.

1991 *Mathematics Subject Classification.* Primary 14F05; Secondary 18E30, 18F20.

* Partially supported by MTM2004-00666.

** Partially supported by MTM2004-00666.

\mathcal{E} on \mathbb{P}^2 with rank r and Chern classes c_1 and c_2 ; and to describe moduli spaces of stable vector bundles on \mathbb{P}^2 . In the succeeding papers [8], [9], [11], [14] and [23], the notion of exceptional bundle on \mathbb{P}^2 was extended to other manifolds X and even more from the category of vector bundles on X to the bounded derived category of coherent sheaves on X .

Any exceptional collection $(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_m)$ gives rise to a bi-infinite collection, $\{\mathcal{E}_i\}_{i \in \mathbb{Z}}$, called *helix* and defined recursively by left and right mutations (see Definition 2.12). Helix theory was introduced by Drezet and Le Potier in [7] and by Gorodentsev and Rudakov in [11], in connection with the problem of constructing exceptional bundles on \mathbb{P}^n ; and helix theory got its further progress in the succeeding papers [2], [6], [9], [18], [19], [22] and [23]. Again helix theory was first developed for vector bundles on \mathbb{P}^2 and generalized later to any triangulated category where for any two objects \mathcal{E} and \mathcal{F} , $\text{Hom}^\bullet(\mathcal{E}, \mathcal{F})$ has a structure of finite dimensional graded vector space over \mathbb{C} .

In this paper, we make an effort to link the abstract and general context of helix theory and exceptional collections to concrete examples, their applications, and the geometrical properties that we can derive. First of all, we recall the notion of geometric collection (see Definition 2.15) introduced by Bondal and Polishchuk in [3]. It is well known that the length of any full exceptional collection of coherent sheaves $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_m)$ on a smooth projective variety X of dimension n is equal to the rank of the Grothendieck group $K_0(X)$ which turns out to be greater or equal to $n+1$. We call geometric collection any full exceptional collection of coherent sheaves of length $n+1$. Geometric collections have nice properties: They are automatically full strongly exceptional collections, their strong exceptionality is preserved under mutations, any thread of an helix associated to a geometric collection is a full strongly exceptional collection, etc...

We address the problem of determining smooth projective varieties with geometric collections and we prove that \mathbb{P}^n , any quadric hypersurface $Q_n \subset \mathbb{P}^{n+1}$ (n odd) and any Fano 3-fold X with $\text{Pic}(X) \cong \mathbb{Z}$ and trivial intermediate Jacobian have geometric collections. Given a coherent sheaf \mathcal{F} on a smooth projective variety X with a geometric collection, we derive two spectral sequences: A Beilinson-Kapranov type spectral sequence which converges to \mathcal{F} (Theorem 2.21) and an Eilenberg-Moore type spectral sequence which abuts to the cohomology of \mathcal{F} (Theorem 2.22).

The existence of geometric collections allows us to generalize the notion of Castelnuovo-Mumford regularity for coherent sheaves on projective spaces to coherent sheaves on smooth projective varieties with a geometric collection (Definition 3.1). We also prove that many of the main properties of the Castelnuovo-Mumford regularity are accomplished by this new concept. Finally, given a smooth projective variety $X \xrightarrow{i} \mathbb{P}^n$ with a geometric collection σ and a coherent \mathcal{O}_X -module \mathcal{F} it would be very interesting to compare the regularity of \mathcal{F} with respect to σ , $\text{Reg}_\sigma(\mathcal{F})$, to the Castelnuovo-Mumford regularity $\text{Reg}^{CM}(i_*\mathcal{F})$ of its extension by zero $i_*\mathcal{F}$. In the last part of this work, we will address this problem and we will show by carefully analyzing the cases of coherent sheaves on quadric hypersurfaces $Q_n \subset \mathbb{P}^{n+1}$ (n odd) that, in general, they are very different and we write down a formula which relates both notions of regularity (Theorem 4.3).

Next we outline the structure of this paper. In section 2, we overview the basic facts on exceptional collections, geometric collections, mutations and helix theory; we give examples to illustrate all these concepts, and we describe the spectral sequences that we use in the sequel to develop the theory of regularity with respect to a geometric collection. In section 3, we give, using a Beilinson-Kapranov type spectral sequence, the promised definition of regularity with respect to a geometric collection, we prove that the Castelnuovo-Mumford regularity of a coherent sheaf \mathcal{F} on \mathbb{P}^n coincides with the regularity of \mathcal{F} with respect to a suitable geometric collection of coherent sheaves on \mathbb{P}^n and we show that the main basic properties of the Castelnuovo-Mumford regularity carry over to the new setting. In section 4, we consider odd dimensional quadric hypersurfaces $Q_n \subset \mathbb{P}^{n+1}$ and their geometric collection $\sigma = (\mathcal{O}_{Q_n}, \mathcal{O}_{Q_n}(1), \dots, \mathcal{O}_{Q_n}(n-1), \Sigma(n-1))$, where Σ is the Spinor bundle. We compute the right dual basis of any thread of the strict helix $\mathcal{H}_\sigma = \{\mathcal{E}_i\}_{i \in \mathbb{Z}}$ associated to σ , we illustrate our results on the regularity of a sheaf with respect to σ for the case of sheaves \mathcal{F} on Q_n and we compare it with the regularity of its extension by zero in the embedding $Q_n \subset \mathbb{P}^{n+1}$. This last results show that, in general, the regularity with respect to σ of a coherent sheaf \mathcal{F} on a smooth projective variety $X \hookrightarrow \mathbb{P}^m$ does not square with the Castelnuovo-Mumford regularity of its extension by zero $i_*\mathcal{F}$ in \mathbb{P}^m . We end the paper in §5 with some final comments and questions which naturally arise from this paper.

2. GEOMETRIC COLLECTIONS AND SPECTRAL SEQUENCES

Let X be a smooth projective variety defined over the complex numbers \mathbb{C} and let $\mathcal{D} = D^b(\mathcal{O}_X\text{-mod})$ be the derived category of bounded complexes of coherent sheaves of \mathcal{O}_X -modules. For any pair of objects $A, B \in \text{Obj}(\mathcal{D})$ we introduce the following notation:

$$\text{Hom}^\bullet(A, B) := \bigoplus_{k \in \mathbb{Z}} \text{Ext}_{\mathcal{D}}^k(A, B)$$

and if V^\bullet is a graded vector space and A an object of \mathcal{D} , then the tensor product can be constructed as

$$(2.1) \quad V^\bullet \otimes A = \bigoplus_{\alpha} V^\alpha \otimes A[-\alpha].$$

We will use the dualization defined for graded vector spaces by the rule

$$(2.2) \quad (V^{\times \bullet})_p = (V^\bullet)_{-p}.$$

A covariant cohomological functor Cov^\bullet is called linear if for any β satisfies

$$(2.3) \quad \text{Cov}^\beta(V^\bullet \otimes A) = \bigoplus_{\alpha} V^\alpha \otimes \text{Cov}^\beta(A[-\alpha]) = \bigoplus_{\alpha} V^\alpha \otimes \text{Cov}^{\beta-\alpha}(A).$$

Definition 2.1. Let X be a smooth projective variety.

(i) An object $\mathcal{F} \in \mathcal{D}$ is **exceptional** if $\text{Hom}^\bullet(\mathcal{F}, \mathcal{F})$ is a 1-dimensional algebra generated by the identity.

(ii) An ordered collection $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_m)$ of objects of \mathcal{D} is an **exceptional collection** if each object \mathcal{F}_i is exceptional and $\text{Ext}_{\mathcal{D}}^\bullet(\mathcal{F}_k, \mathcal{F}_j) = 0$ for $j < k$.

(iii) An exceptional collection $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_m)$ of objects of \mathcal{D} is a **strongly exceptional collection** if in addition $\text{Ext}_{\mathcal{D}}^i(\mathcal{F}_j, \mathcal{F}_k) = 0$ for $i \neq 0$ and $j \leq k$.

(iv) An ordered collection of objects of \mathcal{D} , $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_m)$, is a **full (strongly) exceptional collection** if it is a (strongly) exceptional collection $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_m)$ and $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_m$ generate the bounded derived category \mathcal{D} .

Example 2.2. (1) $(\mathcal{O}_{\mathbb{P}^r}, \mathcal{O}_{\mathbb{P}^r}(1), \mathcal{O}_{\mathbb{P}^r}(2), \dots, \mathcal{O}_{\mathbb{P}^r}(r))$ and $(\mathcal{O}_{\mathbb{P}^r}, \Omega_{\mathbb{P}^r}^1(1), \Omega_{\mathbb{P}^r}^2(2), \dots, \Omega_{\mathbb{P}^r}^r(r))$ are full strongly exceptional collections of coherent sheaves on \mathbb{P}^r .

(2) Let $Gr(k, n)$ be the Grassmannian of k -dimensional subspaces of the n -dimensional vector space and let \mathcal{S} be the tautological k -dimensional bundle on X . Denote by $\Sigma^\alpha \mathcal{S}$ the space of the irreducible representations of the group $GL(\mathcal{S})$ with highest weight $\alpha = (\alpha_1, \dots, \alpha_s)$ and $|\alpha| = \sum_{i=1}^s \alpha_i$. Denote by $A(k, n)$ the set of locally free sheaves $\Sigma^\alpha \mathcal{S}$ on $Gr(k, n)$ where α runs over Young diagrams fitting inside a $k \times (n - k)$ rectangle. Set $\rho(k, n) := \sharp A(k, n)$. By [14]; Proposition 2.2 (a) and Proposition 1.4, $A(k, n)$ can be totally ordered in such a way that we obtain a full strongly exceptional collection $(E_1, \dots, E_{\rho(k, n)})$ of locally free sheaves on $Gr(k, n)$.

(3) Let $Q_n \subset \mathbb{P}^{n+1}$, $n \geq 2$, be the quadric hypersurface. By [15]; Proposition 4.9, if n is even and Σ_1, Σ_2 are the Spinor bundles on Q_n , then

$$(\Sigma_1(-n), \Sigma_2(-n), \mathcal{O}_{Q_n}(-n+1), \dots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n})$$

is a full strongly exceptional collection of locally free sheaves on Q_n ; and if n is odd and Σ is the Spinor bundle on Q_n , then

$$(\Sigma(-n), \mathcal{O}_{Q_n}(-n+1), \dots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n})$$

is a full strongly exceptional collection of locally free sheaves on Q_n .

Remark 2.3. The existence of a full strongly exceptional collection $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_m)$ of coherent sheaves on a smooth projective variety X imposes a rather strong restriction on X , namely that the Grothendieck group $K_0(X) = K_0(\mathcal{O}_X - \text{mod})$ is isomorphic to \mathbb{Z}^{m+1} .

Definition 2.4. Let X be a smooth projective variety and let (A, B) be an exceptional pair of objects of \mathcal{D} . We define the **left mutation** of B , $L_A B$, and the **right mutation** of A , $R_B A$, with the aid of the following distinguished triangles in the category \mathcal{D} :

$$(2.4) \quad L_A B \rightarrow \text{Hom}^\bullet(A, B) \otimes A \rightarrow B \rightarrow L_A B[1]$$

$$(2.5) \quad R_B A[-1] \rightarrow A \rightarrow \text{Hom}^{\times \bullet}(A, B) \otimes B \rightarrow R_B A.$$

Remark 2.5. If we apply $\text{Hom}^\bullet(A, \cdot)$ to the triangle (2.4) and we apply $\text{Hom}^\bullet(\cdot, B)$ to the triangle (2.5) we get the following orthogonality relations:

$$\text{Hom}^\bullet(A, L_A B) = 0 \text{ and } \text{Hom}^\bullet(R_B A, B) = 0.$$

Definition 2.6. Let X be a smooth projective variety and let $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_n)$ be an exceptional collection of objects of \mathcal{D} . A **left mutation** (resp. **right mutation**) of σ is defined as follows: for any $1 \leq i \leq n$ a left mutation L_i replaces the i -th pair of consequent

elements $(\mathcal{E}_{i-1}, \mathcal{E}_i)$ by $(L_{\mathcal{E}_{i-1}}\mathcal{E}_i, \mathcal{E}_{i-1})$ and a right mutation R_i replaces the same pair of consequent elements $(\mathcal{E}_{i-1}, \mathcal{E}_i)$ by $(\mathcal{E}_i, R_{\mathcal{E}_i}\mathcal{E}_{i-1})$:

$$L_i\sigma = L_{\mathcal{E}_{i-1}}\sigma = (\mathcal{E}_0, \dots, L_{\mathcal{E}_{i-1}}\mathcal{E}_i, \mathcal{E}_{i-1}, \dots, \mathcal{E}_n)$$

$$R_i\sigma = R_{\mathcal{E}_{i-1}}\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_i, R_{\mathcal{E}_i}\mathcal{E}_{i-1}, \dots, \mathcal{E}_n).$$

Notation 2.7. It is convenient to agree that

$$R^{(j)}\mathcal{E}_i = R^{(j-1)}R\mathcal{E}_i = R_{\mathcal{E}_{i+j}} \cdots R_{\mathcal{E}_{i+2}}R_{\mathcal{E}_{i+1}}\mathcal{E}_i$$

and similar notation for compositions of left mutations. According to these notations, mutations satisfy the following relations:

$$\begin{aligned} L_i R_i &= R_i L_i = Id \\ L_i L_j &= L_j L_i \quad \text{for } |i - j| > 1 \\ L_{i+1} L_i L_{i+1} &= L_i L_{i+1} L_i \quad \text{for } 1 < i < n. \end{aligned}$$

Proposition 2.8. *Let X be a smooth projective variety and let $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_n)$ be an exceptional collection of objects of \mathcal{D} . Then any mutation of σ is an exceptional collection and if σ generates the category \mathcal{D} , then the mutated collection also generates \mathcal{D} .*

Proof. See [2]; Assertion 2.1 and Lemma 2.2. □

Remark 2.9. In general a mutation of a strongly exceptional collection is not a strongly exceptional collection. For instance, let $X = \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric surface in \mathbb{P}^3 and denote by $\mathcal{O}_X(a, b) = \mathcal{O}_{\mathbb{P}^1}(a) \boxtimes \mathcal{O}_{\mathbb{P}^1}(b)$. By [4]; Proposition 4.16, (see also [5])

$$\sigma = (\mathcal{O}_X, \mathcal{O}_X(1, 0), \mathcal{O}_X(0, 1), \mathcal{O}_X(1, 1))$$

is a full strongly exceptional collection of line bundles on X . Using the exact sequence

$$0 \rightarrow \mathcal{O}_X(-1, 1) \rightarrow \text{Hom}(\mathcal{O}_X(0, 1), \mathcal{O}_X(1, 1)) \otimes \mathcal{O}_X(0, 1) \rightarrow \mathcal{O}_X(1, 1) \rightarrow 0$$

we get that $L_{\mathcal{O}_X(0, 1)}\mathcal{O}_X(1, 1) = \mathcal{O}_X(-1, 1)$. But, since $\text{Ext}^1(\mathcal{O}_X(1, 0), \mathcal{O}_X(-1, 1)) = \mathbb{C}$ the mutated exceptional collection of line bundles on X

$$L\sigma = (\mathcal{O}_X, \mathcal{O}_X(1, 0), \mathcal{O}_X(-1, 1), \mathcal{O}_X(0, 1))$$

is no more a strongly exceptional collection of line bundles on X . We will come back to the problem of whether strongly exceptionality is preserved under mutations.

Definition 2.10. Let X be a smooth projective variety. Given any full exceptional collection $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_n)$ the collection

$$(L^{(n)}\mathcal{E}_n, L^{(n-1)}\mathcal{E}_{n-1}, \dots, L^{(1)}\mathcal{E}_1, \mathcal{E}_0)$$

will be called **left dual base of σ** and the collection

$$(\mathcal{E}_n, R^{(1)}\mathcal{E}_{n-1}, \dots, R^{(n)}\mathcal{E}_0)$$

will be called **right dual base of σ** .

Remark 2.11. Given a full exceptional collection $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_n)$, its corresponding right and left dual basis are uniquely determined up to a unique isomorphism and they satisfy the following orthogonality conditions:

$$\mathrm{Hom}^\alpha(R^{(j)}\mathcal{E}_i, \mathcal{E}_k) = 0, \quad \mathrm{Hom}^\alpha(\mathcal{E}_k, L^{(j)}\mathcal{E}_i) = 0$$

for all α, i, j and k except

$$\mathrm{Hom}^k(R^{(k)}\mathcal{E}_{n-k}, \mathcal{E}_{n-k}) = \mathrm{Hom}^{n-k}(\mathcal{E}_{n-k}, L^{(n-k)}\mathcal{E}_{n-k}) = \mathbb{C}.$$

The definition of helix and the first results about helices appeared in [7] and [11]. Let us recall its definition.

Definition 2.12. Let X be a smooth projective variety. A **helix of period $n + 1$** is an infinite sequence $\{\mathcal{E}_i\}_{i \in \mathbb{Z}}$ of objects of \mathcal{D} such that for any $i \in \mathbb{Z}$, $(\mathcal{E}_i, \dots, \mathcal{E}_{i+n})$ is an exceptional collection and $\mathcal{E}_{n+1+i} = R^{(n)}\mathcal{E}_i$.

Any exceptional collection of objects of \mathcal{D} , $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_n)$, induces a unique helix by the rule

$$(2.6) \quad \mathcal{E}_{n+i} = R^{(n)}\mathcal{E}_{i-1} \quad \text{and} \quad \mathcal{E}_{-i} = L^{(n)}\mathcal{E}_{n-i+1}, \quad i > 0.$$

In that case we say that the helix is generated by σ and that the collection $\mathcal{H}_\sigma := \{\mathcal{E}_i\}_{i \in \mathbb{Z}}$ is the helix associated to σ . Each collection $\sigma_i = (\mathcal{E}_i, \mathcal{E}_{i+1}, \dots, \mathcal{E}_{i+n})$ is called a **thread of the helix** and it is clear that a helix is generated by any of its thread.

Example 2.13. Let $\sigma = (\mathcal{O}_{\mathbb{P}^r}, \mathcal{O}_{\mathbb{P}^r}(1), \mathcal{O}_{\mathbb{P}^r}(2), \dots, \mathcal{O}_{\mathbb{P}^r}(r))$ be the full exceptional collection of line bundles on $\mathbb{P}^r = \mathbb{P}(V)$ given in Example 2.2, (1). The helix associated to σ is given by $\mathcal{H}_\sigma = \{\mathcal{O}_{\mathbb{P}^r}(i)\}_{i \in \mathbb{Z}}$. Indeed, denote by $\mathcal{E}_i = \mathcal{O}_{\mathbb{P}^r}(i)$, $0 \leq i \leq r$. By definition, for any $i > 0$, $\mathcal{E}_{r+i} = R^{(r)}\mathcal{E}_{i-1}$ and $\mathcal{E}_{-i} = L^{(r)}\mathcal{E}_{r-i+1}$. Using the exterior powers of the Euler sequence

$$(2.7) \quad 0 \longrightarrow \wedge^{k-1}T_{\mathbb{P}^r} \longrightarrow \wedge^k V \otimes \mathcal{O}_{\mathbb{P}^r}(k) \longrightarrow \wedge^k T_{\mathbb{P}^r} \longrightarrow 0$$

we deduce that for any $k > 0$

$$R^{(k)}\mathcal{E}_0 = R_{\mathcal{E}_k} R_{\mathcal{E}_{k-1}} \cdots R_{\mathcal{E}_1} \mathcal{E}_0 = \wedge^k T_{\mathbb{P}^r}.$$

So, in particular we get $\mathcal{E}_{r+1} = R^{(r)}\mathcal{E}_0 = \wedge^r T_{\mathbb{P}^r} = \mathcal{O}_{\mathbb{P}^r}(r+1)$. More in general, repeating the process and using once more the exact sequence (2.7) we get

$$\mathcal{E}_{r+i} = R^{(r)}\mathcal{E}_{i-1} = \wedge^r T_{\mathbb{P}^r}(i-1) = \mathcal{O}_{\mathbb{P}^r}(r+i).$$

Analogously, we deduce that $\mathcal{E}_{-i} = \mathcal{O}_{\mathbb{P}^r}(-i)$ and hence we get that the helix associated to σ is given by $\mathcal{H}_\sigma = \{\mathcal{O}_{\mathbb{P}^r}(i)\}_{i \in \mathbb{Z}}$. Notice that in this case, any thread $\sigma_i = (\mathcal{O}_{\mathbb{P}^r}(i), \mathcal{O}_{\mathbb{P}^r}(i+1), \mathcal{O}_{\mathbb{P}^r}(i+2), \dots, \mathcal{O}_{\mathbb{P}^r}(i+r))$ of the helix is again a full strongly exceptional collection of line bundles. Moreover, we have that for any $i \in \mathbb{Z}$,

$$\mathcal{E}_i = \mathcal{E}_{i+r+1} \otimes K_{\mathbb{P}^r},$$

where $K_{\mathbb{P}^r} = \mathcal{O}_{\mathbb{P}^r}(-r-1)$ is the canonical line bundle.

This last observation is indeed a more general fact:

Remark 2.14. Let X be a smooth projective variety of dimension m and let $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_n)$ be a full exceptional collection of objects of \mathcal{D} . Then the helix \mathcal{H}_σ associated to σ has the following property of periodicity: for any $i \in \mathbb{Z}$,

$$\mathcal{E}_i = \mathcal{E}_{i+n+1} \otimes K_X[m-n]$$

where K_X is the canonical line bundle on X and the number in square brackets denotes the multiplicity of the shift of an object to the left viewed as a graded complex in \mathcal{D} .

Bondal and Polishchuk introduced in [3] the notion of geometric collection as the exceptional collection σ of objects of \mathcal{D} that generates a geometric helix, i.e., an helix $\mathcal{H}_\sigma = \{\mathcal{E}_i\}_{i \in \mathbb{Z}}$ such that for any $k > 0$ and any $i \leq j$, $\text{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) = 0$. Then they proved that full geometric collections are exactly full exceptional collections of length equal to the dimension of the variety plus one (see [3]; Proposition 3.3). We find it more convenient to use this latter property as the definition:

Definition 2.15. Let X be a smooth projective variety of dimension n . We call **geometric collection** to any full exceptional collection of coherent sheaves on X of length $n+1$ and we call **strict helix** to the helix generated by it.

Remark 2.16. (1) Notice that since all full strongly exceptional collections of coherent sheaves on X have the same length and it is equal to the $\text{rank}(K_0(X)) \geq n+1$, the length of a geometric collection is the minimum possible.

(2) The existence of a geometric collection on a smooth variety X imposes a strong restriction on X , namely that its Grothendieck group $K_0(X) \cong \mathbb{Z}^{\dim X + 1}$ and X is forced to be a Fano variety (see [3]; Theorem 3.4).

Geometric collections have a nice behavior. For instance, geometric collections are automatically strongly exceptional collections of coherent sheaves and its strongly exceptionality is preserved under mutations (Recall that, in general, strongly exceptionality is not preserved under mutations as we have showed in Remark 2.9). More precisely we have:

Proposition 2.17. *Let X be a smooth projective variety of dimension n and let $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_n)$ be a geometric collection of coherent sheaves on X . Then,*

- (i) *Any mutation of the collection σ consists also of sheaves, i.e. complexes concentrated in the zero component of the grading.*
- (ii) *The collection σ is a full strongly exceptional collection of coherent sheaves.*
- (iii) *Any mutation of σ is a full strongly exceptional collection of coherent sheaves.*
- (iv) *Any thread $(\mathcal{E}_i, \mathcal{E}_{i+1}, \dots, \mathcal{E}_{n+i})$ of the helix \mathcal{H}_σ associated to σ is a full strongly exceptional collection of coherent sheaves on X .*

Proof. It follows from [2]; Assertion 9.2, Theorem 9.3 and Corollary 9.4. \square

Example 2.18. According to Example 2.2, there exists a geometric collection of coherent sheaves on \mathbb{P}^r and on $Q_n \subset \mathbb{P}^{n+1}$, n odd, and hence both varieties have strict helices. On the other hand, there are no geometric collections of coherent sheaves on Q_n for n even and on $Gr(k, n) = Gr(k, V)$, $2 \leq k \leq n-2$.

Proposition 2.19. *Let X be any smooth Fano threefold with $\text{Pic}(X) \cong \mathbb{Z}$ and trivial intermediate Jacobian. Then, X has a geometric collection.*

Proof. According to the classification of Fano threefolds X with $\text{Pic}(X) \cong \mathbb{Z}$ and trivial intermediate Jacobian ([13]; Table 3.5), there exist four kinds of such manifolds: The projective space \mathbb{P}^3 , a smooth quadric $Q_3 \subset \mathbb{P}^4$, the manifold $V_5 \subset \mathbb{P}^6$ and the family of manifolds $V_{22} \subset \mathbb{P}^{12}$. The cases $X \cong \mathbb{P}^3$ and $X \cong Q_3$ follow from Example 2.18, (1) and (2), respectively. The case $X \cong V_5$ is due to Orlov ([20]) and the case $X \cong V_{22}$ is due to Kuznetsov ([16]; Theorem 3). \square

Remark 2.20. Restricting ourselves to the strict helix area \mathcal{H}_σ , $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_n)$ all the theory can be pulled down from the triangulated category $\mathcal{D} = D^b(\mathcal{O}_X\text{-mod})$ of bounded complexes of coherent sheaves into the category of coherent sheaves $\text{Coh}(X)$. For instance, for any $i < j$ the canonical distinguished triangles

$$(2.8) \quad L_{\mathcal{E}_i} \mathcal{E}_j \rightarrow \text{Hom}^\bullet(\mathcal{E}_i, \mathcal{E}_j) \otimes \mathcal{E}_i \rightarrow \mathcal{E}_j \rightarrow L_{\mathcal{E}_i} \mathcal{E}_j[1]$$

$$(2.9) \quad R_{\mathcal{E}_j} \mathcal{E}_i[-1] \rightarrow \mathcal{E}_i \rightarrow \text{Hom}^{\times\bullet}(\mathcal{E}_i, \mathcal{E}_j) \otimes \mathcal{E}_j \rightarrow R_{\mathcal{E}_j} \mathcal{E}_i,$$

turn to usual triples of coherent sheaves

$$(2.10) \quad 0 \rightarrow L_{\mathcal{E}_i} \mathcal{E}_j \rightarrow \text{Hom}(\mathcal{E}_i, \mathcal{E}_j) \otimes \mathcal{E}_i \rightarrow \mathcal{E}_j \rightarrow 0$$

$$(2.11) \quad 0 \rightarrow \mathcal{E}_i \rightarrow \text{Hom}^*(\mathcal{E}_i, \mathcal{E}_j) \otimes \mathcal{E}_j \rightarrow R_{\mathcal{E}_j} \mathcal{E}_i \rightarrow 0.$$

So, without loss of generality, we could define the mutations inside a strict helix in terms of sheaves.

Theorem 2.21. (Beilinson-Kapranov type spectral sequence) *Let X be a smooth projective variety of dimension n with a geometric collection $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$ and let $\mathcal{H}_\sigma = \{\mathcal{E}_i\}_{i \in \mathbb{Z}}$ be the associated strict helix. Then for any thread $\sigma_i = (\mathcal{E}_i, \dots, \mathcal{E}_{i+n})$ of the helix \mathcal{H}_σ and any coherent sheaf \mathcal{F} on X there is a spectral sequence with E_1 -term*

$$(2.12) \quad {}^i E_1^{pq} = \text{Ext}^q(R^{(-p)} \mathcal{E}_{i+n+p}, \mathcal{F}) \otimes \mathcal{E}_{i+p+n}$$

situated in the square $0 \leq q \leq n$, $-n \leq p \leq 0$ which converges to

$$E_\infty^i = \begin{cases} \mathcal{F} & \text{for } i = 0 \\ 0 & \text{for } i \neq 0. \end{cases}$$

Proof. First of all notice that since the helix \mathcal{H}_σ is strict, the thread $\sigma_i = (\mathcal{E}_i, \dots, \mathcal{E}_{i+n})$ also generates the category \mathcal{D} . We write V_k^\bullet for the graded vector spaces

$$V_k^\bullet = \text{Hom}^\bullet(R^{(n-k)} \mathcal{E}_{k+i}, \mathcal{F})$$

and we consider the complex

$$L^\bullet : 0 \rightarrow V_0^\bullet \otimes \mathcal{E}_i \rightarrow V_1^\bullet \otimes \mathcal{E}_{i+1} \rightarrow \dots \rightarrow V_{n-1}^\bullet \otimes \mathcal{E}_{i+n-1} \rightarrow V_n^\bullet \otimes \mathcal{E}_{i+n} \rightarrow 0$$

where the tensor product is defined as in (2.1). The right mutations produce a canonical right Postnikov system of the complex L^\bullet , which naturally identifies \mathcal{F} with the canonical

right convolution of this complex. Then, for an arbitrary linear covariant cohomological functor Φ^\bullet , there exists an spectral sequence with E_1 -term

$${}^i E_1^{pq} = \Phi^q(L^p)$$

situated in the square $0 \leq p, q \leq n$ and converging to $\Phi^{p+q}(\mathcal{F})$ (see [15]; 1.5). Since Φ^\bullet is a linear functor, it follows from (2.3) that

$$(2.13) \quad \Phi^q(L^p) = \Phi^q(V_p^\bullet \otimes \mathcal{E}_{i+p}) = \bigoplus_l V_p^l \otimes \Phi^{q-l}(\mathcal{E}_{i+p}) = \bigoplus_{\alpha+\beta=q} V_p^\alpha \otimes \Phi^\beta(\mathcal{E}_{i+p}).$$

In particular, if we consider the covariant linear cohomology functor which takes a complex to its cohomology sheaves and acts identically on sheaves, i.e.

$$\Phi^\beta(\mathcal{F}) = \begin{cases} \mathcal{F} & \text{for } \beta = 0 \\ 0 & \text{for } \beta \neq 0 \end{cases}$$

on any sheaf \mathcal{F} , in the square $0 \leq p, q \leq n$, we get

$${}^i E_1^{pq} = V_p^q \otimes \mathcal{E}_{i+p} = \text{Ext}^q(R^{(n-p)} \mathcal{E}_{p+i}, \mathcal{F}) \otimes \mathcal{E}_{i+p}$$

where the last equality follows from the definition of V_k^\bullet . Finally, if we call $p' = p - n$ we get the spectral sequence with E_1 -term

$${}^i E_1^{p'q} = \text{Ext}^q(R^{(-p')} \mathcal{E}_{p'+n+i}, \mathcal{F}) \otimes \mathcal{E}_{p'+n+i}$$

in the square $0 \leq q \leq n, -n \leq p' \leq 0$ and which converges to

$$E_\infty^i = \begin{cases} \mathcal{F} & \text{for } i = 0 \\ 0 & \text{for } i \neq 0. \end{cases}$$

□

Theorem 2.22. (Eilenberg-Moore type spectral sequence) *Let X be a smooth projective variety of dimension n with a geometric collection $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$ and let $\mathcal{H}_\sigma = \{\mathcal{E}_i\}_{i \in \mathbb{Z}}$ be the associated strict helix. Then for any thread $\sigma_i = (\mathcal{E}_i, \dots, \mathcal{E}_{i+n})$ of the helix \mathcal{H}_σ and any pair of coherent sheaves \mathcal{F}, \mathcal{G} on X there is a spectral sequence with E_1 -term*

$$(2.14) \quad {}^i E_1^{pq} = \bigoplus_{\alpha+\beta=q} \text{Ext}^\alpha(R^{(-p)} \mathcal{E}_{i+n+p}, \mathcal{F}) \otimes \text{Ext}^\beta(\mathcal{G}, \mathcal{E}_{i+p+n})$$

situated in the square $0 \leq q \leq n, -n \leq p \leq 0$ which converges to

$$E_\infty^i = \begin{cases} \text{Ext}^{p+q}(\mathcal{G}, \mathcal{F}) & \text{for } i = 0 \\ 0 & \text{for } i \neq 0. \end{cases}$$

Proof. We follow step by step the proof of Theorem 2.21 but in this case in (2.13) we take as Φ^\bullet the covariant functor $\text{Hom}^\bullet(\mathcal{G}, \cdot)$ and in such a way we get the spectral sequence

$${}^i E_1^{pq} = \bigoplus_{\alpha+\beta=q} \text{Ext}^\alpha(R^{(-p)} \mathcal{E}_{i+n+p}, \mathcal{F}) \otimes \text{Ext}^\beta(\mathcal{G}, \mathcal{E}_{i+p+n})$$

situated in the square $0 \leq q \leq n, -n \leq p \leq 0$ which converges to $\Phi^{p+q}(\mathcal{F}) = \text{Ext}^{p+q}(\mathcal{G}, \mathcal{F})$. □

We will end this section with technical results that will be used in next sections.

Lemma 2.23. *Let X be a smooth projective variety of dimension n and let $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_n)$ be a geometric collection of coherent sheaves on X . For any $i < j$ and any invertible sheaf \mathcal{F} , it holds:*

- (a) $(L_{\mathcal{E}_i} \mathcal{E}_j)^* = R_{\mathcal{E}_i^*} \mathcal{E}_j^*$;
- (b) $(R_{\mathcal{E}_j} \mathcal{E}_i)^* = L_{\mathcal{E}_j^*} \mathcal{E}_i^*$;
- (c) $(R_{\mathcal{E}_j} \mathcal{E}_i) \otimes \mathcal{F} \cong R_{\mathcal{E}_j \otimes \mathcal{F}}(\mathcal{E}_i \otimes \mathcal{F})$ and $(L_{\mathcal{E}_i} \mathcal{E}_j) \otimes \mathcal{F} \cong L_{\mathcal{E}_i \otimes \mathcal{F}}(\mathcal{E}_j \otimes \mathcal{F})$.

Proof. (a) According to Remark 2.20; (2.10), $L_{\mathcal{E}_i} \mathcal{E}_j$ is given by the aid of the exact sequence

$$(2.15) \quad 0 \rightarrow L_{\mathcal{E}_i} \mathcal{E}_j \rightarrow \text{Hom}(\mathcal{E}_i, \mathcal{E}_j) \otimes \mathcal{E}_i \rightarrow \mathcal{E}_j \rightarrow 0.$$

Dualizing this exact sequence we get

$$0 \rightarrow \mathcal{E}_j^* \rightarrow \text{Hom}(\mathcal{E}_j^*, \mathcal{E}_i^*) \otimes \mathcal{E}_i^* \cong \text{Hom}^*(\mathcal{E}_i, \mathcal{E}_j) \otimes \mathcal{E}_i^* \rightarrow (L_{\mathcal{E}_i} \mathcal{E}_j)^* \rightarrow 0$$

which according to Remark 2.20; (2.11) gives that $(L_{\mathcal{E}_i} \mathcal{E}_j)^* = R_{\mathcal{E}_i^*} \mathcal{E}_j^*$.

(b) The proof is analogous to the proof of (a).

(c) Since $\text{Hom}(\mathcal{E}_i, \mathcal{E}_j) \cong \text{Hom}(\mathcal{E}_i \otimes \mathcal{F}, \mathcal{E}_j \otimes \mathcal{F})$, it is enough to tensor by \mathcal{F} the exact sequences

$$0 \rightarrow L_{\mathcal{E}_i} \mathcal{E}_j \rightarrow \text{Hom}(\mathcal{E}_i, \mathcal{E}_j) \otimes \mathcal{E}_i \rightarrow \mathcal{E}_j \rightarrow 0$$

and

$$0 \rightarrow \mathcal{E}_i \rightarrow \text{Hom}^*(\mathcal{E}_i, \mathcal{E}_j) \otimes \mathcal{E}_j \rightarrow R_{\mathcal{E}_j} \mathcal{E}_i \rightarrow 0.$$

□

Corollary 2.24. *Let X be a smooth projective variety of dimension n with canonical line bundle K and let $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_n)$ be a geometric collection of coherent sheaves on X . Assume that $\tau = (\mathcal{F}_0, \dots, \mathcal{F}_n)$ is the right dual base of σ . Then, for any integer λ , the right dual base of $\sigma_{\lambda(n+1)} = (\mathcal{E}_{\lambda(n+1)}, \mathcal{E}_{\lambda(n+1)+1}, \dots, \mathcal{E}_{\lambda(n+1)+n})$ is*

$$\tau_{\lambda(n+1)} = (\mathcal{F}_0 \otimes (K^*)^{\otimes \lambda}, \mathcal{F}_1 \otimes (K^*)^{\otimes \lambda}, \dots, \mathcal{F}_n \otimes (K^*)^{\otimes \lambda}).$$

Proof. By definition of right dual base we have

$$\mathcal{F}_j = R^{(j)} \mathcal{E}_{n-j} \quad 0 \leq j \leq n.$$

On the other hand, by Remark 2.14

$$\mathcal{E}_{\lambda(n+1)+i} = \mathcal{E}_i \otimes (K^*)^{\otimes \lambda}.$$

Therefore, applying Lemma 2.23, we get

$$R^{(j)} \mathcal{E}_{\lambda(n+1)+n-j} = R^{(j)} (\mathcal{E}_{n-j} \otimes (K^*)^{\otimes \lambda}) = (R^{(j)} \mathcal{E}_{n-j}) \otimes (K^*)^{\otimes \lambda} = \mathcal{F}_j \otimes (K^*)^{\otimes \lambda}.$$

□

3. REGULARITY WITH RESPECT TO GEOMETRIC COLLECTIONS: DEFINITION AND PROPERTIES

In this section, using the so-called Beilinson-Kapranov spectral sequence, we generalize the notion of Castelnuovo-Mumford regularity for coherent sheaves on a projective space to coherent sheaves on a smooth projective variety with a geometric collection of coherent sheaves. We establish for coherent sheaves on \mathbb{P}^n the agreement of the new definition of regularity with the old one and we prove that many formal properties of Castelnuovo-Mumford regularity continue to hold in our more general setup.

Let X be a smooth projective variety of dimension n and let $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$ be a geometric collection on X . Associated to σ we have a strict helix $\mathcal{H}_\sigma = \{\mathcal{E}_i\}_{i \in \mathbb{Z}}$; and for any collection $\sigma_i = (\mathcal{E}_i, \mathcal{E}_{i+1}, \dots, \mathcal{E}_{i+n})$ of $n+1$ subsequent sheaves (i.e. for any thread of the helix \mathcal{H}_σ) and any coherent \mathcal{O}_X -module \mathcal{F} a spectral sequence (See Theorem 2.21)

$$(3.1) \quad {}^i E_1^{pq} = \text{Ext}^q(R^{(-p)}\mathcal{E}_{i+p+n}, \mathcal{F}) \otimes \mathcal{E}_{i+p+n}$$

situated in the square $0 \leq q \leq n, -n \leq p \leq 0$ which converges to

$$E_\infty^i = \begin{cases} \mathcal{F} & \text{for } i = 0 \\ 0 & \text{for } i \neq 0. \end{cases}$$

Definition 3.1. Let X be a smooth projective variety of dimension n with a geometric collection $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$ and let \mathcal{F} be a coherent \mathcal{O}_X -module. We say that \mathcal{F} is **m -regular with respect to σ** if $\text{Ext}^q(R^{(-p)}\mathcal{E}_{-m+p}, \mathcal{F}) = 0$ for $q > 0$ and $-n \leq p \leq 0$.

So, \mathcal{F} is m -regular with respect to σ if ${}^{-n-m}E_1^{pq} = 0$ for $q > 0$ in (3.1). In particular, if \mathcal{F} is m -regular with respect to σ the spectral sequence ${}^{-n-m}E_1^{pq}$ collapses at E_2 and we get the following exact sequence:

$$(3.2) \quad 0 \longrightarrow \mathcal{L}_{-n} \longrightarrow \dots \longrightarrow \mathcal{L}_{-1} \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{L}_p = H^0(X, (R^{(-p)}\mathcal{E}_{-m+p})^* \otimes \mathcal{F}) \otimes \mathcal{E}_{-m+p}$ for $-n \leq p \leq 0$.

Definition 3.2. Let X be a smooth projective variety of dimension n with a geometric collection $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$ and let \mathcal{F} be a coherent \mathcal{O}_X -module. We define the **regularity of \mathcal{F} with respect to σ** , $\text{Reg}_\sigma(\mathcal{F})$, as the least integer m such that \mathcal{F} is m -regular with respect to σ . We set $\text{Reg}_\sigma(\mathcal{F}) = -\infty$ if there is no such integer.

Remark 3.3. It would be nice to characterize the sheaves \mathcal{F} on X with $\text{Reg}_\sigma(\mathcal{F}) = -\infty$.

Example 3.4. Let V be a \mathbb{C} -vector space of dimension $n+1$ and set $\mathbb{P}^n = \mathbb{P}(V)$. We consider the geometric collection $\sigma = (\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n))$ on \mathbb{P}^n and the associated strict helix $\mathcal{H}_\sigma = \{\mathcal{O}_{\mathbb{P}^n}(i)\}_{i \in \mathbb{Z}}$ (see Example 2.13). Using the exterior powers of the Euler sequence

$$0 \longrightarrow \wedge^{k-1} T_{\mathbb{P}^n} \longrightarrow \wedge^k V \otimes \mathcal{O}_{\mathbb{P}^n}(k) \longrightarrow \wedge^k T_{\mathbb{P}^n} \longrightarrow 0$$

we compute the right dual basis of any thread $\sigma_i = (\mathcal{O}_{\mathbb{P}^n}(i), \mathcal{O}_{\mathbb{P}^n}(i+1), \dots, \mathcal{O}_{\mathbb{P}^n}(i+n))$ of the helix \mathcal{H}_σ and we get

$$\begin{aligned} & (\mathcal{O}_{\mathbb{P}^n}(n+i), R^{(1)}\mathcal{O}_{\mathbb{P}^n}(i+n-1), \dots, R^{(j)}\mathcal{O}_{\mathbb{P}^n}(i+n-j), \dots, R^{(n)}\mathcal{O}_{\mathbb{P}^n}(i)) \\ &= (\mathcal{O}_{\mathbb{P}^n}(n+i), T_{\mathbb{P}^n}(i+n-1), \dots, \wedge^j T_{\mathbb{P}^n}(i+n-j), \dots, \wedge^n T_{\mathbb{P}^n}(i)). \end{aligned}$$

Therefore, for any coherent sheaf \mathcal{F} on \mathbb{P}^n our definition reduces to say: \mathcal{F} is m -regular with respect to σ if $\text{Ext}^q(\wedge^{-p}T(-m+p), \mathcal{F}) = H^q(\mathbb{P}^n, \Omega^{-p}(m-p) \otimes \mathcal{F}) = 0$ for all $q > 0$ and all $p, -n \leq p \leq 0$.

We now compute the regularity with respect to $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$ of the sheaves \mathcal{E}_i .

Proposition 3.5. *Let X be a smooth projective variety of dimension n with a geometric collection $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$ and let $\mathcal{H}_\sigma = \{\mathcal{E}_i\}_{i \in \mathbb{Z}}$ be the associated strict helix. Then, for any $i \in \mathbb{Z}$, $\text{Reg}_\sigma(\mathcal{E}_i) = -i$.*

Proof. First of all we will see that $\text{Reg}_\sigma(\mathcal{E}_i) \leq -i$. By Remark 2.5, we have

$$\text{Ext}^q(R^{(-p)}\mathcal{E}_{i+p}, \mathcal{E}_i) = \text{Ext}^q(R_{\mathcal{E}_i} \cdots R_{\mathcal{E}_{i+p+2}} R_{\mathcal{E}_{i+p+1}} \mathcal{E}_{i+p}, \mathcal{E}_i) = 0$$

for $q > 0$ and $-n \leq p \leq 0$. So, \mathcal{E}_i is $(-i)$ -regular with respect to σ or, equivalently, $\text{Reg}_\sigma(\mathcal{E}_i) \leq -i$.

Let us now see that \mathcal{E}_i is not $(-i-1)$ -regular with respect to σ . To this end, it is enough to see that there is $q > 0$ and there is $p, -n \leq p \leq 0$, such that

$$\text{Ext}^q(R^{(-p)}\mathcal{E}_{i+1+p}, \mathcal{E}_i) \neq 0.$$

To prove it, we write $i = \alpha n + j$ with $0 \leq j < n$, $\alpha \in \mathbb{Z}$, we consider the thread $\sigma_{\alpha n} = (\mathcal{E}_{\alpha n}, \mathcal{E}_{\alpha n+1}, \dots, \mathcal{E}_{\alpha n+j} = \mathcal{E}_i, \dots, \mathcal{E}_{\alpha n+n})$ and we construct its right dual basis (see Definition 2.10)

$$(\mathcal{E}_{\alpha n+n}, R^{(1)}\mathcal{E}_{\alpha n+n-1}, \dots, R^{(n-j)}\mathcal{E}_{\alpha n+j}, \dots, R^{(n)}\mathcal{E}_{\alpha n}).$$

It follows from Remark 2.11 that

$$\text{Ext}^{n-j}(R^{(n-j)}\mathcal{E}_{\alpha n+j}, \mathcal{E}_{\alpha n+j}) = \text{Ext}^{n-j}(R_{\mathcal{E}_{\alpha n+n}} R_{\mathcal{E}_{\alpha n+n-1}} \cdots R_{\mathcal{E}_{\alpha n+j+1}} \mathcal{E}_{\alpha n+j}, \mathcal{E}_{\alpha n+j}) = \mathbb{C}.$$

We consider the exact sequence

$$\begin{aligned} 0 \longrightarrow R_{\mathcal{E}_{\alpha n+n-1}} \cdots R_{\mathcal{E}_{\alpha n+j+1}} \mathcal{E}_{\alpha n+j} &\longrightarrow \text{Hom}^*(R_{\mathcal{E}_{\alpha n+n-1}} \cdots R_{\mathcal{E}_{\alpha n+j+1}} \mathcal{E}_{\alpha n+j}, \mathcal{E}_{\alpha n+n}) \otimes \mathcal{E}_{\alpha n+n} \\ &\longrightarrow R_{\mathcal{E}_{\alpha n+n}} R_{\mathcal{E}_{\alpha n+n-1}} \cdots R_{\mathcal{E}_{\alpha n+j+1}} \mathcal{E}_{\alpha n+j} \longrightarrow 0 \end{aligned}$$

and we apply the contravariant functor $\text{Hom}(\cdot, \mathcal{E}_{\alpha n+j})$. Since $\text{Ext}^q(\mathcal{E}_{\alpha n+n}, \mathcal{E}_{\alpha n+j}) = 0$ for $q > 0$, we get

$$\text{Ext}^{n-j-1}(R_{\mathcal{E}_{\alpha n+n-1}} \cdots R_{\mathcal{E}_{\alpha n+j+1}} \mathcal{E}_{\alpha n+j}, \mathcal{E}_{\alpha n+j}) = \mathbb{C}.$$

We repeat the process using the consequent right mutations and we get

$$\text{Ext}^{n-j-k}(R_{\mathcal{E}_{\alpha n+n-k}} \cdots R_{\mathcal{E}_{\alpha n+j+1}} \mathcal{E}_{\alpha n+j}, \mathcal{E}_{\alpha n+j}) = \mathbb{C}$$

for $0 \leq k \leq n-1-j$. In particular,

$$\text{Ext}^1(R_{\mathcal{E}_{i+1}} \mathcal{E}_i, \mathcal{E}_i) = \mathbb{C}$$

which implies that \mathcal{E}_i is not $(-i-1)$ -regular and we conclude that $\text{Reg}_\sigma(\mathcal{E}_i) = -i$. \square

In [17], Lecture 14, D. Mumford defined the notion of regularity for a coherent sheaf over a projective space. Let us recall it.

Definition 3.6. A coherent sheaf \mathcal{F} on \mathbb{P}^n is said to be *m -regular in the sense of Castelnuovo-Mumford* if $H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0$ for $i > 0$. We define the Castelnuovo-Mumford regularity of \mathcal{F} , $\text{Reg}^{CM}(\mathcal{F})$, as the least integer m such that \mathcal{F} is m -regular.

Notice that such an m always exists by the ampleness of $\mathcal{O}_{\mathbb{P}^n}(1)$ ([12]; Chap. III, Proposition 5.3).

Let us now establish for coherent sheaves on \mathbb{P}^n the agreement of regularity definition in the sense of Definition 3.1 with Castelnuovo-Mumford definition.

Proposition 3.7. *A coherent sheaf \mathcal{F} on $\mathbb{P}^n = \mathbb{P}(V)$ is m -regular in the sense of Castelnuovo-Mumford if and only if it is m -regular with respect to the geometric collection $\sigma = (\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n))$. Hence, we have*

$$\text{Reg}_\sigma(\mathcal{F}) = \text{Reg}^{CM}(\mathcal{F}).$$

Proof. According to Definitions 3.1 and 3.6, and Example 3.4 we have to see that

$$(3.3) \quad H^q(\mathbb{P}^n, \mathcal{F}(m-q)) = 0 \text{ for all } q > 0$$

if and only if

$$(3.4) \quad H^q(\mathbb{P}^n, \mathcal{F} \otimes \Omega_{\mathbb{P}^n}^p(m+p)) = 0 \text{ for all } q > 0 \text{ and } 0 \leq p \leq n.$$

Let us first see that (3.3) implies (3.4). By [17], Lecture 14, the equalities (3.3) are equivalent to

$$(3.5) \quad H^q(\mathbb{P}^n, \mathcal{F}(t)) = 0 \text{ for all } q > 0 \text{ and } t \geq m - q.$$

Since $\Omega_{\mathbb{P}^n}^n(n) \cong \mathcal{O}_{\mathbb{P}^n}(-1)$, we deduce from (3.5) that

$$(3.6) \quad H^q(\mathbb{P}^n, \mathcal{F} \otimes \Omega_{\mathbb{P}^n}^n(n+t)) = 0 \text{ for all } q > 0 \text{ and all } t \geq m - q + 1.$$

Using the exact cohomology sequence

$$(3.7) \quad \dots \longrightarrow H^q(\mathbb{P}^n, V \otimes \mathcal{F}(t)) \longrightarrow H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n-1}(n-1) \otimes \mathcal{F}(t+1)) \longrightarrow H^{q+1}(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^n(n) \otimes \mathcal{F}(t)) \longrightarrow \dots$$

associated to

$$0 \longrightarrow \Omega_{\mathbb{P}^n}^n(n) \longrightarrow \wedge^n V \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \Omega_{\mathbb{P}^n}^{n-1}(n) \longrightarrow 0$$

together with the equalities (3.6) and (3.5) we deduce

$$(3.8) \quad H^q(\mathbb{P}^n, \mathcal{F} \otimes \Omega_{\mathbb{P}^n}^{n-1}(n-1+t)) = 0 \text{ for all } q > 0 \text{ and all } t \geq m - q + 1.$$

Going on using the exact cohomology sequence

$$(3.9) \quad \dots \longrightarrow H^q(\mathbb{P}^n, \wedge^{i+1} V \otimes \mathcal{F}(t)) \longrightarrow H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^i(i) \otimes \mathcal{F}(t+1)) \longrightarrow H^{q+1}(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{i+1}(i+1) \otimes \mathcal{F}(t)) \longrightarrow \dots$$

associated to

$$0 \longrightarrow \Omega_{\mathbb{P}^n}^{i+1}(i+1) \longrightarrow \wedge^{i+1} V \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \Omega_{\mathbb{P}^n}^i(i+1) \longrightarrow 0$$

we deduce

$$(3.10) \quad H^q(\mathbb{P}^n, \mathcal{F} \otimes \Omega_{\mathbb{P}^n}^i(i+t)) = 0 \text{ for all } q > 0 \text{ and all } t \geq m - q + 1$$

which obviously implies (3.4). Let us prove the converse. To this end we consider the Eilenberg-Moore type spectral sequence (see Theorem 2.22)

$$E_1^{pq} = \bigoplus_{\alpha+\beta=q} \left(\text{Ext}^\alpha(R^{(-p)}\mathcal{E}_{i+n+p}, \mathcal{F}) \otimes \text{Ext}^\beta(\mathcal{G}, \mathcal{E}_{i+n+p}) \right) \quad 0 \leq q \leq n, \quad -n \leq p \leq 0$$

$$d_r^{pq} : E_r^{pq} \longrightarrow E_r^{p+r, q-r+1}, \quad E_\infty \Rightarrow \text{Ext}^{p+q}(\mathcal{G}, \mathcal{F})$$

and we apply it to the case $i = -m - n$ and $\mathcal{G} = \mathcal{O}_{\mathbb{P}^n}(t - m)$. So, we have

$$E_1^{pq} = \bigoplus_{\alpha+\beta=q} \left(H^\alpha(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{(-p)} \otimes \mathcal{F}) \otimes H^\beta(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p - t)) \right) \quad 0 \leq q \leq n, \quad -n \leq p \leq 0.$$

By (3.4), $H^\alpha(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{(-p)} \otimes \mathcal{F}) = 0$ for all $\alpha > 0$ and all p , $-n \leq p \leq 0$ and by Bott's formulas a non-zero cohomology group of line bundles on \mathbb{P}^n , $H^\beta(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p - t))$, corresponds only to $\beta = 0$ or n and in the latter case $p - t \leq -n - 1$ (i.e. $p \leq t - n - 1$). Thus, $E_1^{pq} \neq 0$ forces $p + q \leq t - 1$. Therefore, $E_\infty^{pq} = 0$ for $p + q \geq t$ and so $0 = \text{Ext}^t(\mathcal{O}_{\mathbb{P}^n}(t - m), \mathcal{F}) = H^t(\mathbb{P}^n, \mathcal{F}(m - t))$ for $t \geq 1$ or, equivalently, \mathcal{F} is m -regular in the sense of Castelnuovo-Mumford. \square

Let us now prove that the main formal properties of Castelnuovo-Mumford regularity over projective spaces remain to be true in the new setting.

Proposition 3.8. *Let X be a smooth projective variety of dimension n with a geometric collection $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$ and let \mathcal{F} be a coherent \mathcal{O}_X -module. If \mathcal{F} is m -regular with respect to σ then the canonical map $\text{Hom}(\mathcal{E}_{-m}, \mathcal{F}) \otimes \mathcal{E}_{-m} \twoheadrightarrow \mathcal{F}$ is surjective and \mathcal{F} is k -regular with respect to σ for any $k \geq m$ as well.*

Proof. The first assertion immediately follows from the exact sequence (3.2) of \mathcal{F} . To prove the second assertion, it is enough to check it for $k = m + 1$. Since \mathcal{F} is m -regular with respect to σ we have

$$(3.11) \quad \text{Ext}^q(R^{(-p)}\mathcal{E}_{-m+p}, \mathcal{F}) = 0 \text{ for all } q > 0 \text{ and all } -n \leq p \leq 0.$$

In order to prove that \mathcal{F} is $(m + 1)$ -regular with respect to σ we have to check that

$$(3.12) \quad \text{Ext}^q(R^{(-p)}\mathcal{E}_{-m-1+p}, \mathcal{F}) = 0 \text{ for all } q > 0 \text{ and all } -n \leq p \leq 0.$$

To this end we apply the contravariant functor $\text{Hom}(\cdot, \mathcal{F})$ to the exact sequence

$$0 \longrightarrow \mathcal{E}_{-m-1} \longrightarrow \text{Hom}^*(\mathcal{E}_{-m-1}, \mathcal{E}_{-m}) \otimes \mathcal{E}_{-m} \longrightarrow R_{\mathcal{E}_{-m}}\mathcal{E}_{-m-1} \longrightarrow 0$$

and we get the exact sequence

$$\begin{aligned} \cdots \longrightarrow \text{Ext}^q(R_{\mathcal{E}_{-m}}\mathcal{E}_{-m-1}, \mathcal{F}) \longrightarrow \text{Hom}^*(\mathcal{E}_{-m-1}, \mathcal{E}_{-m}) \otimes \text{Ext}^q(\mathcal{E}_{-m}, \mathcal{F}) \longrightarrow \\ \text{Ext}^q(\mathcal{E}_{-m-1}, \mathcal{F}) \longrightarrow \text{Ext}^{q+1}(R_{\mathcal{E}_{-m}}\mathcal{E}_{-m-1}, \mathcal{F}) \longrightarrow \cdots \end{aligned}$$

Since by (3.11), $\text{Ext}^q(\mathcal{E}_{-m}, \mathcal{F}) = \text{Ext}^q(R_{\mathcal{E}_{-m}}\mathcal{E}_{-m-1}, \mathcal{F}) = 0$ for all $q > 0$, we get that $\text{Ext}^q(\mathcal{E}_{-m-1}, \mathcal{F}) = 0$ for all $q > 0$. Using again (3.11) and the exact sequence

$$0 \longrightarrow R_{\mathcal{E}_{-m-1}} \mathcal{E}_{-m-2} \longrightarrow \text{Hom}^*(R_{\mathcal{E}_{-m-1}} \mathcal{E}_{-m-2}, \mathcal{E}_{-m}) \otimes \mathcal{E}_{-m} \longrightarrow R_{\mathcal{E}_{-m}} R_{\mathcal{E}_{-m-1}} \mathcal{E}_{-m-2} \longrightarrow 0$$

we get that $\text{Ext}^q(R_{\mathcal{E}_{-m-1}} \mathcal{E}_{-m-2}, \mathcal{F}) = 0$ for all $q > 0$.

Going on and using the consequent right mutations, we get for all i , $1 \leq i \leq n-1$,

$$\text{Ext}^q(R_{\mathcal{E}_{-m-1}} R_{\mathcal{E}_{-m-2}} \cdots R_{\mathcal{E}_{-m-i}} \mathcal{E}_{-m-i-1}, \mathcal{F}) = 0 \text{ for all } q > 0.$$

Hence, it only remains to see that

$$\text{Ext}^q(R_{\mathcal{E}_{-m-1}} R_{\mathcal{E}_{-m-2}} \cdots R_{\mathcal{E}_{-m-n}} \mathcal{E}_{-m-n-1}, \mathcal{F}) = 0 \text{ for all } q > 0.$$

The vanishing of these last Ext's groups follows again from (3.11) taking into account that, by Definition 2.10

$$R_{\mathcal{E}_{-m-1}} R_{\mathcal{E}_{-m-2}} \cdots R_{\mathcal{E}_{-m-n}} \mathcal{E}_{-m-n-1} = R^{(n)} \mathcal{E}_{-m-n-1} = \mathcal{E}_{-m}.$$

□

Proposition 3.9. *Let X be a smooth projective variety of dimension n with a geometric collection $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$, let \mathcal{F} and \mathcal{G} be coherent \mathcal{O}_X -modules and let*

$$(3.13) \quad 0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

be an exact sequence of coherent \mathcal{O}_X -modules. Then,

- (a) $\text{Reg}_\sigma(\mathcal{F}_2) \leq \max\{\text{Reg}_\sigma(\mathcal{F}_1), \text{Reg}_\sigma(\mathcal{F}_3)\}.$
- (b) $\text{Reg}_\sigma(\mathcal{F} \oplus \mathcal{G}) = \max\{\text{Reg}_\sigma(\mathcal{F}), \text{Reg}_\sigma(\mathcal{G})\}.$

Proof. (a) Let $m = \max\{\text{Reg}_\sigma(\mathcal{F}_1), \text{Reg}_\sigma(\mathcal{F}_3)\}$. Since, by Proposition 3.8, \mathcal{F}_1 and \mathcal{F}_3 are both m -regular with respect to σ considering the long exact sequence

$$\cdots \rightarrow \text{Ext}^q(R^{(-p)} \mathcal{E}_{-m+p}, \mathcal{F}_1) \rightarrow \text{Ext}^q(R^{(-p)} \mathcal{E}_{-m+p}, \mathcal{F}_2) \rightarrow \text{Ext}^q(R^{(-p)} \mathcal{E}_{-m+p}, \mathcal{F}_3) \rightarrow \cdots$$

associated to (3.13) we get $\text{Ext}^q(R^{(-p)} \mathcal{E}_{-m+p}, \mathcal{F}_2) = 0$ for any $q > 0$ and $-n \leq p \leq 0$, which implies that $\text{Reg}_\sigma(\mathcal{F}_2) \leq m$.

(b) It is enough to see that if \mathcal{F} is m -regular with respect to σ and \mathcal{G} is s -regular with respect to σ then $\mathcal{F} \oplus \mathcal{G}$ is $t = \max(s, m)$ -regular with respect to σ . Since, by Proposition 3.8, \mathcal{F} and \mathcal{G} are both t -regular with respect to σ and, moreover, the functor Ext's is additive we get

$$\text{Ext}^q(R^{(-p)} \mathcal{E}_{-t+p}, \mathcal{F} \oplus \mathcal{G}) = \text{Ext}^q(R^{(-p)} \mathcal{E}_{-t+p}, \mathcal{F}) = \text{Ext}^q(R^{(-p)} \mathcal{E}_{-t+p}, \mathcal{G}) = 0$$

for $q > 0$ and $-n \leq p \leq 0$. Therefore, $\mathcal{F} \oplus \mathcal{G}$ is t -regular with respect to σ . □

The following Example will show that in Proposition 3.9, (a) we can have strict inequality.

Example 3.10. We consider the geometric collection $\sigma = (\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n))$ of locally free sheaves on \mathbb{P}^n and we denote by H a hyperplane section on \mathbb{P}^n . For any $n \geq r \geq 0$, we have the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(r-1) \rightarrow \mathcal{O}_{\mathbb{P}^n}(r) \rightarrow \mathcal{O}_H(r) \rightarrow 0.$$

According to Proposition 3.5, $\text{Reg}_\sigma(\mathcal{O}_{\mathbb{P}^n}(r)) = -r$ and $\text{Reg}_\sigma(\mathcal{O}_{\mathbb{P}^n}(r-1)) = -r+1$. So, $\text{Reg}_\sigma(\mathcal{O}_{\mathbb{P}^n}(r)) < \max\{\text{Reg}_\sigma(\mathcal{O}_{\mathbb{P}^n}(r-1)), \text{Reg}_\sigma(\mathcal{O}_H(r))\}$.

In Proposition 3.7, we have seen that a coherent sheaf \mathcal{F} on \mathbb{P}^n is m -regular in the sense of Castelnuovo-Mumford if and only if it is m -regular with respect to the geometric collection $\sigma = (\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n))$. A more subtle problem is the following one. Let $X \xrightarrow{i} \mathbb{P}^n$ be a smooth projective variety, let \mathcal{F} be a coherent sheaf on X and let $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_n)$ be a geometric collection of coherent sheaves on X . How are related the regularity of \mathcal{F} with respect to σ and the Castelnuovo-Mumford regularity of its extension by zero $i_*\mathcal{F}$ via the embedding $X \xrightarrow{i} \mathbb{P}^n$? In next section, we will address this problem for the case of coherent sheaves on odd dimensional quadric hypersurfaces $Q_n \subset \mathbb{P}^{n+1}$.

4. REGULARITY OF SHEAVES ON Q_n

Let $n \in \mathbb{Z}$ be an odd integer ($n = 2t + 1$) and let $Q_n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface. In [15], M. M. Kapranov defined the locally free sheaves ψ_i , $i \geq 0$, on Q_n and the Spinor bundle Σ on Q_n to construct a resolution of the diagonal $\Delta \subset Q_n \times Q_n$ and to describe the bounded derived category $D^b(\mathcal{O}_{Q_n} - \text{mod})$. In particular, he got that

$$(\Sigma(-n), \mathcal{O}_{Q_n}(-n+1), \dots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n})$$

is a full strongly exceptional collection of locally free sheaves on Q_n ([15]; Proposition 4.9) and hence, according to Definition 2.15, Q_n has a geometric collection of locally free sheaves. Dualising each bundle of the above geometric sequence and reversing the order, we get that

$$\sigma := (\mathcal{O}_{Q_n}, \mathcal{O}_{Q_n}(1), \dots, \mathcal{O}_{Q_n}(n-1), \Sigma(n-1))$$

is also a geometric collection of locally free sheaves on Q_n .

In this section we will give an elementary description of the locally free sheaves ψ_i and their basic properties (for more details the reader can look at [15]) and, for any coherent sheaf \mathcal{F} on Q_n , we will relate $\text{Reg}_\sigma(\mathcal{F})$ to the Castelnuovo-Mumford regularity of its extension by zero $i_*\mathcal{F}$ in the embedding $i : Q_n \hookrightarrow \mathbb{P}^{n+1}$.

From now on, we set $\Omega^j := \Omega_{\mathbb{P}^{n+1}}^j$ and we define inductively ψ_j :

$$\psi_0 := \mathcal{O}_{Q_n}, \quad \psi_1 := \Omega^1(1)|_{Q_n}$$

and, for all $j \geq 2$, we define the locally free sheaf ψ_j as the unique non-splitting extension (Note that $\text{Ext}^1(\psi_{j-2}, \Omega^j(j)|_{Q_n}) = \mathbb{C}$):

$$0 \longrightarrow \Omega^j(j)|_{Q_n} \longrightarrow \psi_j \longrightarrow \psi_{j-2} \longrightarrow 0.$$

In particular, $\psi_{j+2} = \psi_j$ for $j \geq n$ and $\psi_n = \Sigma(-1)^{2^{t+1}}$, ($n = 2t + 1$).

Before computing the right dual basis of any thread $\sigma_j = (\mathcal{E}_j, \mathcal{E}_{j+1}, \dots, \mathcal{E}_{j+n})$ of the helix $\mathcal{H}_\sigma = \{\mathcal{E}_j\}_{j \in \mathbb{Z}}$ associated to σ , we collect in the following Lemma the cohomological properties of the Spinor bundles we need later.

Lemma 4.1. *Let $n \in \mathbb{Z}$ be an odd integer, let $Q_n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface and let Σ be the Spinor bundle on Q_n . Then,*

(i) $H^i(Q_n, \Sigma(t)) = 0$ for any i such that $0 < i < n$ and for all $t \in \mathbb{Z}$.

- (ii) $H^0(Q_n, \Sigma(t)) = 0$ for all $t < 0$ and $h^0(Q_n, \Sigma) = 2^{\frac{n+1}{2}}$.
- (iii) $\text{Ext}^i(\Sigma(j), \Sigma) = \begin{cases} 0 & \text{if } i \neq j \\ \mathbb{C} & \text{if } i = j \end{cases}$ for any $1 \leq j \leq n$.
- (iv) $H^i(Q_n, \psi_j(-j)) = \begin{cases} 0 & \text{if } i \neq j \\ \mathbb{C} & \text{if } i = j \end{cases}$ for any $1 \leq j \leq n$.

Proof. The assertions (i) and (ii) follows from [21]; Theorem 2.3. The assertion (iii) follows by induction on j , using (i) and (ii) and the long exact sequence

$$\cdots \rightarrow \text{Ext}^{i-1}(\mathcal{O}_{Q_n}(j)^{2^{\frac{n+1}{2}}}, \Sigma) \rightarrow \text{Ext}^{i-1}(\Sigma(j-1), \Sigma) \rightarrow \text{Ext}^i(\Sigma(j), \Sigma) \rightarrow \text{Ext}^i(\mathcal{O}_{Q_n}(j)^{2^{\frac{n+1}{2}}}, \Sigma) \rightarrow \cdots$$

obtained by applying the functor $\text{Hom}(\cdot, \Sigma)$ to the exact sequence

$$0 \rightarrow \Sigma(j-1) \rightarrow \mathcal{O}_{Q_n}(j)^{2^{\frac{n+1}{2}}} \rightarrow \Sigma(j) \rightarrow 0.$$

The assertion (iv) follows from [15]; Proposition 4.11. \square

Proposition 4.2. *Let $n \in \mathbb{Z}$ be an odd integer, let $Q_n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface and let $\mathcal{H}_\sigma = \{\mathcal{E}_i\}_{i \in \mathbb{Z}}$ be the helix associated to*

$$\sigma = (\mathcal{O}_{Q_n}, \mathcal{O}_{Q_n}(1), \dots, \mathcal{O}_{Q_n}(n-1), \Sigma(n-1)).$$

Let us denote by σ_k the thread $(\mathcal{E}_k, \mathcal{E}_{k+1}, \dots, \mathcal{E}_{k+n})$. Then:

(a) *The right dual base of σ_0 is*

$$(\Sigma(n-1), \psi_{n-1}(n), \psi_{n-2}(n), \dots, \psi_1(n), \psi_0(n)).$$

(b) *For any j , $1 \leq j \leq n$, the right dual base of the geometric collection*

$$\sigma_j = (\mathcal{O}_{Q_n}(j), \dots, \mathcal{O}_{Q_n}(n-1), \Sigma(n-1), \mathcal{O}_{Q_n}(n), \mathcal{O}_{Q_n}(n+1), \dots, \mathcal{O}_{Q_n}(n+j-1))$$

is

$$(\mathcal{O}_{Q_n}(n+j-1), \psi_1^*(n+j-1), \dots, \psi_{j-1}^*(n+j-1), \Sigma(n+j-1), \psi_{n-j-1}(n+j), \dots, \psi_0(n+j)).$$

(c) *For any $\lambda \in \mathbb{Z}$, the right dual base of the geometric collection*

$$\sigma_{\lambda(n+1)} = (\mathcal{O}_{Q_n}(\lambda n), \mathcal{O}_{Q_n}(1+\lambda n), \dots, \mathcal{O}_{Q_n}(n-1+\lambda n), \Sigma(n-1+\lambda n))$$

is

$$(\Sigma(n-1+\lambda n), \psi_{n-1}((\lambda+1)n), \psi_{n-2}((\lambda+1)n), \dots, \psi_1((\lambda+1)n), \psi_0((\lambda+1)n)).$$

(d) *For any j , $1 \leq j \leq n$ and any $\lambda \in \mathbb{Z}$, the right dual base of the geometric collection*

$$\sigma_{j+\lambda(n+1)} = (\mathcal{O}_{Q_n}(j+\lambda n), \dots, \mathcal{O}_{Q_n}(n-1+\lambda n), \Sigma(n-1+\lambda n), \mathcal{O}_{Q_n}((\lambda+1)n), \dots, \mathcal{O}_{Q_n}(j-2+(\lambda+1)n), \mathcal{O}_{Q_n}(j-1+(\lambda+1)n))$$

is

$$(\mathcal{O}_{Q_n}((\lambda+1)n+j-1), \psi_1^*((\lambda+1)n+j-1), \dots, \psi_{j-1}^*((\lambda+1)n+j-1), \Sigma((\lambda+1)n+j-1), \psi_{n-j-1}((\lambda+1)n+j), \dots, \psi_0((\lambda+1)n+j)).$$

Proof. (a) By [15]; Proposition 4.11 and using the fact that the right dual basis of an exceptional collection is uniquely determined up to unique isomorphism by the orthogonality conditions described in Remark 2.11, we get that the right dual basis of the exceptional collection $(\Sigma(-n), \mathcal{O}_{Q_n}(-n+1), \dots, \mathcal{O}_{Q_n}(-1), \mathcal{O}_{Q_n})$ is

$$(\mathcal{O}_{Q_n}, \psi_1^*, \dots, \psi_{n-1}^*, \Sigma^*(1)).$$

In particular, we have

$$(4.1) \quad R^{(j)}\mathcal{O}_{Q_n}(-j) = \psi_j^* \text{ for } 0 \leq j \leq n-1; \text{ and}$$

$$R^{(n)}\Sigma(-n) = \Sigma^*(1).$$

Let us now compute the right dual base of σ_0 . Since $R^{(0)}\Sigma(n-1) = \Sigma(n-1)$ and $R^{(n)}\mathcal{O}_{Q_n} = \mathcal{O}_{Q_n}(n)$, according to the definition of right dual base we only need to compute $R^{(j)}\mathcal{O}(n-j)$ for $1 \leq j \leq n-1$. It follows from Lemma 2.23 and the fact that $\Sigma^*(-n+1) = \Sigma(-n)$, that

$$\begin{aligned} R^{(j)}\mathcal{O}(n-j) &= R_{\Sigma(n-1)\mathcal{O}_{Q_n}(n-1)\dots\mathcal{O}_{Q_n}(n-j+1)}\mathcal{O}(n-j) \\ &= (L_{\Sigma(-n)\mathcal{O}_{Q_n}(-n+1)\dots\mathcal{O}_{Q_n}(j-n-1)}\mathcal{O}(j-n))^* \\ &= (L_{\Sigma\mathcal{O}_{Q_n}(1)\dots\mathcal{O}_{Q_n}(j-1)}\mathcal{O}(j))^* \otimes \mathcal{O}(n). \end{aligned}$$

By [10]; 2.8.1,

$$\begin{aligned} (L_{\Sigma\mathcal{O}_{Q_n}(1)\dots\mathcal{O}_{Q_n}(j-1)}\mathcal{O}(j))^* \otimes \mathcal{O}(n) &= (R_{\mathcal{O}_{Q_n}(-n+j+1), \mathcal{O}_{Q_n}(-n+j+2)\dots\mathcal{O}_{Q_n}}\mathcal{O}_{Q_n}(-n+j))^* \otimes \mathcal{O}_{Q_n}(n) \\ &= (R^{(j)}\mathcal{O}_{Q_n}(-j))^* \otimes \mathcal{O}_{Q_n}(n) \\ &= \psi_{n-j}(n) \end{aligned}$$

where the last equality follows from (4.1). Hence, $R^{(j)}\mathcal{O}(n-j) = \psi_{n-j}(n)$ which finishes the proof of (a).

(b) Applying Lemma 2.23; (c) to (4.1), we get that

$$(4.2) \quad R^{(p)}\mathcal{O}_{Q_n}(n+j-1-p) = \psi_p^*(n+j-1) \quad \text{for } 0 \leq p \leq j-1.$$

On the other hand, by Lemma 4.1 we have:

- For any $\alpha \in \mathbb{Z}$ and $j \leq t \leq n+j-1$

$$\begin{aligned} \text{Ext}^\alpha(\Sigma(n+j-1), \mathcal{O}_{Q_n}(t)) &= 0; \quad \text{and} \\ \text{Ext}^\alpha(\Sigma(n+j-1), \Sigma(n-1)) &= \begin{cases} 0 & \alpha \neq j \\ \mathbb{C} & \alpha = j. \end{cases} \end{aligned}$$

- For any $\alpha \in \mathbb{Z}$, $n-j-1 \leq t \leq 0$ and $j \leq \gamma \leq n+j-1$

$$\begin{aligned} \text{Ext}^\alpha(\psi_t(n+j), \Sigma(n-1)) &= 0; \quad \text{and} \\ \text{Ext}^\alpha(\psi_t(n+j), \mathcal{O}_{Q_n}(\gamma)) &= \begin{cases} \mathbb{C} & \text{if } n-\alpha = t = \gamma-j \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since the right dual base of an exceptional collection is uniquely determined up to unique isomorphism by the orthogonal conditions given in Remark 2.11, it follows from (4.2), these last cohomological relations and from Lemma 4.1 that indeed

$$\begin{aligned} &(\mathcal{O}_{Q_n}(n+j-1), \psi_1^*(n+j-1), \psi_2^*(n+j-1), \dots, \psi_{j-1}^*(n+j-1), \\ &\quad \Sigma(n+j-1), \psi_{n-j-1}(n+j), \dots, \psi_1(n+j), \psi_0(n+j)) \end{aligned}$$

is the right dual base of σ_j .

(c) and (d) follows from (a) and (b), respectively, and Corollary 2.24. \square

Let \mathcal{F} be a coherent sheaf on Q_n and let $i_*\mathcal{F}$ be its extension by zero in the embedding $Q_n \xrightarrow{i} \mathbb{P}^{n+1}$. We are now ready to compare $\text{Reg}_\sigma(\mathcal{F})$ to $\text{Reg}^{CM}(i_*\mathcal{F})$ and state the main result of this section.

Theorem 4.3. *Let n be an odd integer and let $Q_n \xrightarrow{i} \mathbb{P}^{n+1}$ be a quadric hypersurface. We consider the geometric collection $\sigma = (\mathcal{O}_{Q_n}, \dots, \mathcal{O}_{Q_n}(n-1), \Sigma(n-1))$. For any coherent sheaf \mathcal{F} on Q_n we have:*

$$\lfloor \frac{n\text{Reg}_\sigma(\mathcal{F})}{n+1} \rfloor \leq \text{Reg}^{CM}(i_*\mathcal{F}) \leq \lfloor \frac{n\text{Reg}_\sigma(\mathcal{F})}{n+1} \rfloor + 1.$$

Proof. First of all, we will see that $\text{Reg}^{CM}(i_*\mathcal{F}) \leq \lfloor \frac{n\text{Reg}_\sigma(\mathcal{F})}{n+1} \rfloor + 1$. To this end, since for all $t \in \mathbb{Z}$ and $0 \leq q \leq n$,

$$(4.3) \quad H^{n+1}(\mathbb{P}^{n+1}, i_*\mathcal{F}(t)) = 0, \quad H^q(\mathbb{P}^{n+1}, i_*\mathcal{F}(t)) \cong H^q(Q_n, \mathcal{F}(t)),$$

we will see that if \mathcal{F} is m -regular with respect to σ then

$$H^i(Q_n, \mathcal{F}(m + \lambda + 1 - i)) = 0, \quad 1 \leq i \leq n$$

where $\lambda := \lfloor \frac{-m}{n+1} \rfloor$. Write $-m = \lambda(n+1) + r$ with $0 \leq r \leq n$. So,

$$-m - n = \begin{cases} (\lambda - 1)(n + 1) + r + 1 & \text{if } 0 \leq r \leq n - 1 \\ \lambda(n + 1) & \text{if } r = n. \end{cases}$$

Therefore, $\sigma_{-m-n} = (\mathcal{E}_{-m-n}, \dots, \mathcal{E}_{-m})$ is equal to

$$(\mathcal{O}_{Q_n}((\lambda - 1)n + r + 1), \dots, \mathcal{O}_{Q_n}((\lambda - 1)n + n - 1), \Sigma((\lambda - 1)n + n - 1), \mathcal{O}_{Q_n}(\lambda n), \mathcal{O}_{Q_n}(\lambda n + 1), \dots, \mathcal{O}_{Q_n}(\lambda n + r)),$$

if $0 \leq r \leq n - 1$ and is equal to

$$(\mathcal{O}_{Q_n}(\lambda n), \mathcal{O}_{Q_n}(\lambda n + 1), \dots, \mathcal{O}_{Q_n}(\lambda n + n - 1), \Sigma(\lambda n + n - 1))$$

if $r = n$. We consider the Eilenberg-Moore type spectral sequence (see Theorem 2.22)

$$E_1^{pq} = \bigoplus_{\alpha+\beta=q} \left(\text{Ext}^\alpha(R^{(-p)}\mathcal{E}_{i+n+p}, \mathcal{F}) \otimes \text{Ext}^\beta(\mathcal{G}, \mathcal{E}_{i+n+p}) \right) \quad 0 \leq q \leq n, \quad -n \leq p \leq 0$$

$$d_r^{pq} : E_r^{pq} \longrightarrow E_r^{p+r, q-r+1}, \quad E_\infty \Rightarrow \text{Ext}^{p+q}(\mathcal{G}, \mathcal{F})$$

and we apply it to the case $i = -m - n$ and $\mathcal{G} = \mathcal{O}_{Q_n}(t - m - \lambda - 1)$. So, in the square $0 \leq q \leq n, -n \leq p \leq 0$ we have

$$E_1^{pq} = \bigoplus_{\alpha+\beta=q} \left(\text{Ext}^\alpha(R^{(-p)}\mathcal{E}_{-m+p}, \mathcal{F}) \otimes \text{Ext}^\beta(\mathcal{O}_{Q_n}(t - m - \lambda - 1), \mathcal{E}_{-m+p}) \right).$$

Since \mathcal{F} is m -regular with respect to σ , $\text{Ext}^\alpha(R^{(-p)}\mathcal{E}_{-m+p}, \mathcal{F}) = 0$ for all $\alpha > 0$ and all p , $-n \leq p \leq 0$. On the other hand, if $0 \leq r \leq n-1$, then

$$\text{Ext}^\beta(\mathcal{O}_{Q_n}(t-m-\lambda-1), \mathcal{E}_{-m+p}) = \begin{cases} H^\beta(Q_n, \mathcal{O}_{Q_n}(2-t+p)) & \text{if } -n \leq p \leq -r-2 \\ H^\beta(Q_n, \Sigma(-t+p+1)) & \text{if } p = -r-1 \\ H^\beta(Q_n, \mathcal{O}_{Q_n}(1-t+p)) & \text{if } -r \leq p \leq 0, \end{cases}$$

and if $r = n$, then

$$\text{Ext}^\beta(\mathcal{O}_{Q_n}(t-m-\lambda-1), \mathcal{E}_{-m+p}) = \begin{cases} H^\beta(Q_n, \mathcal{O}_{Q_n}(1-t+p)) & \text{if } -n \leq p \leq -1 \\ H^\beta(Q_n, \Sigma(-t)) & \text{if } p = 0. \end{cases}$$

Therefore, applying Serre's duality and Lemma 4.1, we get that the only non-zero Ext 's groups $\text{Ext}^\beta(\mathcal{O}_{Q_n}(t-m-\lambda-1), \mathcal{E}_{-m+p})$ correspond to $\beta = 0$ or $\beta = n$ and in the latter case $-r \leq p \leq 0$ and $p \leq t-1-n$ or $-n \leq p \leq -r-1$ and $p \leq t-2-n$. Thus, $E_1^{pq} \neq 0$ forces $p+q \leq t-1$. Therefore, $E_\infty^{pq} = 0$ for $p+q \geq t$ and so for any $t > 0$,

$$\text{Ext}^t(\mathcal{O}_{Q_n}(t-m-\lambda-1), \mathcal{F}) = H^t(Q_n, \mathcal{F}(m+\lambda+1-t)) = 0.$$

In order to prove the other inequality, we will see that if $i_*\mathcal{F}$ is $(m+\lambda)$ -regular in the sense of Castelnuovo-Mumford, then \mathcal{F} is m -regular with respect to σ , where $\lambda := \lfloor \frac{-m}{n+1} \rfloor$. If $i_*\mathcal{F}$ is m -regular, then it is also $(m+\gamma)$ -regular for all $\gamma \geq 0$ and we have

$$(4.4) \quad 0 = H^i(\mathbb{P}^{n+1}, i_*\mathcal{F}(t)) = H^i(Q_n, \mathcal{F}(t)), \quad \text{for all } t \geq m+\lambda-i, \quad i > 0.$$

We write $-m = \lambda(n+1) + r$ with $0 \leq r \leq n$. If $r \neq n$ then σ_{-m-n} is given by

$$(\mathcal{O}_{Q_n}((\lambda-1)n+r+1), \dots, \mathcal{O}_{Q_n}((\lambda-1)n+n-1), \Sigma((\lambda-1)n+n-1), \mathcal{O}_{Q_n}(\lambda n), \mathcal{O}_{Q_n}(\lambda n+1), \dots, \mathcal{O}_{Q_n}(\lambda n+r)),$$

and by Proposition 4.2 its right dual base is given by

$$(\mathcal{O}_{Q_n}(\lambda n+r), \psi_1^*(\lambda n+r), \dots, \psi_r^*(\lambda n+r), \Sigma(\lambda n+r), \psi_{n-r-2}(\lambda n+r+1), \dots, \psi_0(\lambda n+r+1)),$$

and if $r = n$, σ_{-m-n} is equal to

$$(\mathcal{O}_{Q_n}(\lambda n), \mathcal{O}_{Q_n}(\lambda n+1), \dots, \mathcal{O}_{Q_n}(\lambda n+n-1), \Sigma(\lambda n+n-1))$$

and by Proposition 4.2 its right dual base is given by

$$(\Sigma(n-1+\lambda n), \psi_{n-1}((\lambda+1)n), \psi_{n-2}((\lambda+1)n), \dots, \psi_1((\lambda+1)n), \psi_0((\lambda+1)n)).$$

So, according to Definition 3.1, we have to see that for any $q > 0$

- (i) $\text{Ext}^q(\mathcal{O}_{Q_n}(\lambda n+r), \mathcal{F}) = H^q(Q_n, \mathcal{F}(m+\lambda)) = 0$
- (ii) $\text{Ext}^q(\psi_\alpha^*(\lambda n+r), \mathcal{F}) = \text{Ext}^q(\psi_\alpha^*, \mathcal{F}(m+\lambda)) = 0 \quad 1 \leq \alpha \leq r$
- (iii) $\text{Ext}^q(\psi_{n-\beta}(\lambda n+r+1), \mathcal{F}) = \text{Ext}^q(\psi_{n-\beta}, \mathcal{F}(m+\lambda-1)) = 0 \quad r+2 \leq \beta \leq n$
- (iv) $\text{Ext}^q(\Sigma(\lambda n+r), \mathcal{F}) = \text{Ext}^q(\Sigma, \mathcal{F}(m+\lambda)) = 0$

if $r \neq n$ and we have to see that

- (v) $\text{Ext}^q(\psi_{n-\alpha}((\lambda+1)n), \mathcal{F}) = \text{Ext}^q(\psi_{n-\alpha}, \mathcal{F}(m+\lambda)) = 0 \quad 1 \leq \alpha \leq n$
- (vi) $\text{Ext}^q(\Sigma(\lambda n+n-1), \mathcal{F}) = \text{Ext}^q(\Sigma, \mathcal{F}(m+\lambda+1)) = 0$

if $r = n$. From (4.4) we immediately get (i). Equalities (ii), (iii) and (v) are consequence of the following stronger results

$$(4.5) \quad \text{Ext}^\alpha(\psi_j, \mathcal{F}(t)) = 0 \quad \text{for any } \alpha > 0, \quad 0 \leq j \leq n, \quad \text{and } t \geq m + \lambda - \alpha,$$

and

$$(4.6) \quad \text{Ext}^\alpha(\psi_j^*, \mathcal{F}(t)) = 0 \quad \text{for any } \alpha > 0, \quad 1 \leq j \leq r, \quad \text{and } t \geq m + \lambda - \alpha.$$

We will prove (4.5) by induction on j . Analogously, we can prove (4.6) and we right it to the reader. If $j = 0$, it follows from (4.4) that $\text{Ext}^\alpha(\psi_0, \mathcal{F}(t)) = H^\alpha(Q_n, \mathcal{F}(t)) = 0$ for all $\alpha > 0$ and $t \geq m + \lambda - \alpha$. Assume $j = 1$. Restricting to Q_n the Euler sequence and applying the functor $\text{Ext}(\cdot, \mathcal{F}(t))$ we get the exact sequence

$$\cdots \rightarrow \text{Ext}^\alpha(\mathcal{O}_{Q_n}^{n+2}, \mathcal{F}(t)) \rightarrow \text{Ext}^\alpha(\psi_1, \mathcal{F}(t)) \rightarrow \text{Ext}^{\alpha+1}(\mathcal{O}_{Q_n}(1), \mathcal{F}(t)) \rightarrow \cdots.$$

By (4.4), $\text{Ext}^{\alpha+1}(\mathcal{O}_{Q_n}(1), \mathcal{F}(t)) = H^{\alpha+1}(Q_n, \mathcal{F}(t-1)) = 0$ for $t-1 \geq m + \lambda - \alpha - 1$ and $\text{Ext}^\alpha(\mathcal{O}_{Q_n}^{n+2}, \mathcal{F}(t)) = 0$ for all $\alpha > 0$ and $t \geq m + \lambda - \alpha$. Hence

$$(4.7) \quad \text{Ext}^\alpha(\psi_1, \mathcal{F}(t)) = 0 \quad \text{for all } \alpha > 0 \quad \text{and } t \geq m + \lambda - \alpha.$$

Before going ahead with the general case, let us prove the following

Claim: For any $p \geq 1$, $\alpha > 0$ and $t \geq m + \lambda - \alpha$,

$$\text{Ext}^\alpha(\Omega^p(p)_{|Q_n}, \mathcal{F}(t)) = 0.$$

Proof of the Claim: We will prove it by induction on p . If $p = 1$, by (4.7),

$$\text{Ext}^\alpha(\Omega^1(1)_{|Q_n}, \mathcal{F}(t)) = \text{Ext}^\alpha(\psi_1, \mathcal{F}(t)) = 0$$

for any $\alpha > 0$ and $t \geq m + \lambda - \alpha$. Assume it holds for p and let us see the case $p + 1$. Applying the functor $\text{Ext}(\cdot, \mathcal{F}(t))$ to the exact sequence

$$0 \rightarrow \Omega^{p+1}(p+1)_{|Q_n} \rightarrow \mathcal{O}_{Q_n}^{\binom{n+2}{p+1}} \rightarrow \Omega^p(p+1)_{|Q_n} \rightarrow 0$$

obtained by restricting to Q_n the p -th exterior power of the dual of the Euler sequence on \mathbb{P}^{n+1} , we get the long exact sequence

$$\cdots \rightarrow \text{Ext}^\alpha(\mathcal{O}_{Q_n}^{\binom{n+2}{p+1}}, \mathcal{F}(t)) \rightarrow \text{Ext}^\alpha(\Omega^{p+1}(p+1)_{|Q_n}, \mathcal{F}(t)) \rightarrow \text{Ext}^{\alpha+1}(\Omega^p(p+1)_{|Q_n}, \mathcal{F}(t)) \rightarrow \cdots.$$

By hypothesis of induction, $\text{Ext}^{\alpha+1}(\Omega^p(p+1)_{|Q_n}, \mathcal{F}(t)) = 0$ for any $t-1 \geq m + \lambda - \alpha - 1$ and by (4.4), $\text{Ext}^\alpha(\mathcal{O}_{Q_n}^{\binom{n+2}{p+1}}, \mathcal{F}(t)) = 0$ for any $t \geq m + \lambda - \alpha$. Hence, $\text{Ext}^\alpha(\Omega^{p+1}(p+1)_{|Q_n}, \mathcal{F}(t)) = 0$ for any $t \geq m + \lambda - \alpha$ and $\alpha > 0$ which finishes the proof of the Claim.

Let us now prove (4.5) by induction on j . The first two cases have been already done. Assume (4.5) holds for $j \geq 1$ and let us see that it holds for $j + 1$. Applying the functor $\text{Ext}(\cdot, \mathcal{F}(t))$ to the exact sequence

$$0 \longrightarrow \Omega^j(j)_{|Q_n} \longrightarrow \psi_j \longrightarrow \psi_{j-2} \longrightarrow 0$$

we get the long exact sequence

$$\cdots \rightarrow \text{Ext}^\alpha(\psi_{j-1}, \mathcal{F}(t)) \rightarrow \text{Ext}^\alpha(\psi_{j+1}, \mathcal{F}(t)) \rightarrow \text{Ext}^\alpha(\Omega^{j+1}(j+1)_{|Q_n}, \mathcal{F}(t)) \rightarrow \cdots.$$

It follows from the Claim that $\text{Ext}^\alpha(\Omega^{j+1}(j+1)|_{Q_n}, \mathcal{F}(t)) = 0$ for all $\alpha > 0$ and $t \geq m + \lambda - \alpha$ and by hypothesis of induction $\text{Ext}^\alpha(\psi_{j-1}, \mathcal{F}(t)) = 0$ for all $\alpha > 0$ and $t \geq m + \lambda - \alpha$. Hence we obtain

$$\text{Ext}^\alpha(\psi_{j+1}, \mathcal{F}(t)) = 0$$

for any $\alpha > 0$ and $t \geq m + \lambda - \alpha$, which finishes the proof of (ii), (iii) and (v). Finally, using the fact that $\psi_n = \Sigma(-1)^{2^{t+1}}$, we deduce from (4.5) that (iv) and (vi) also hold. \square

Now we will show that Theorem 4.3 is optimal, i.e., both inequalities can be realized. First, we will compare $\text{Reg}_\sigma(\mathcal{E}_j)$ to $\text{Reg}^{CM}(i_*\mathcal{E}_j)$ where \mathcal{E}_j is part of the helix $\mathcal{H}_\sigma = \{\mathcal{E}_j\}_{j \in \mathbb{Z}}$ associated to σ and later we will compare $\text{Reg}_\sigma(\psi_1(3 + \lambda n))$ to $\text{Reg}^{CM}(i_*(\psi_1(3 + \lambda n)))$.

Proposition 4.4. *Let n be an odd integer and let $i : Q_n \hookrightarrow \mathbb{P}^{n+1}$ be a quadric hypersurface. Let $\mathcal{H}_\sigma = \{\mathcal{E}_j\}_{j \in \mathbb{Z}}$ be the strict helix associated to $\sigma = (\mathcal{O}_{Q_n}, \dots, \mathcal{O}_{Q_n}(n-1), \Sigma(n-1))$. For any integer $j \in \mathbb{Z}$, we write $j = \lambda(n+1) + r$ with $0 \leq r \leq n$. Then, we have*

$$\begin{aligned} \text{Reg}_\sigma(\mathcal{E}_j) &= -j \\ \text{Reg}^{CM}(i_*\mathcal{E}_j) &= -j + \lambda + 1. \end{aligned}$$

In particular, the following relation holds:

$$\text{Reg}_\sigma(\mathcal{E}_j) + \lambda + 1 = \text{Reg}^{CM}(i_*\mathcal{E}_j).$$

Proof. By remark 2.14, if $0 \leq r \leq n-1$, we have

$$\mathcal{E}_j = \mathcal{E}_{\lambda(n+1)+r} = \mathcal{E}_r \otimes (K^*)^{\otimes \lambda} = \mathcal{O}_{Q_n}(\lambda n + r)$$

and if $r = n$, then we have

$$\mathcal{E}_j = \mathcal{E}_{\lambda(n+1)+n} = \mathcal{E}_n \otimes (K^*)^{\otimes \lambda} = \Sigma(\lambda n + n - 1).$$

Using the exact sequence

$$0 \rightarrow \mathcal{O}_{Q_n}(-2) \rightarrow \mathcal{O}_{Q_n} \rightarrow i_*\mathcal{O}_{Q_n} \rightarrow 0$$

we get that

$$\begin{aligned} H^q(\mathbb{P}^{n+1}, i_*\mathcal{O}_{Q_n}(t)) &= 0 \quad \text{for all } t \in \mathbb{Z}, \quad 1 \leq q \leq n-1 \quad \text{and } q = n+1 \\ H^n(\mathbb{P}^{n+1}, i_*\mathcal{O}_{Q_n}(t)) &= 0 \quad \text{for all } t \geq -n+1, \end{aligned}$$

and we conclude that $\text{Reg}^{CM}(i_*\mathcal{O}_{Q_n}(s)) = -s + 1$. In particular, if $j = \lambda(n+1) + r$ with $0 \leq r \leq n-1$, we have

$$\begin{aligned} \text{Reg}^{CM}(i_*\mathcal{E}_j) &= \text{Reg}^{CM}(i_*\mathcal{O}_{Q_n}(\lambda n + r)) \\ &= -\lambda n - r + 1 \\ &= -j + \lambda + 1 \\ &= \text{Reg}_\sigma(\mathcal{E}_j) + \lambda + 1 \end{aligned}$$

where the last equality follows from Proposition 3.5. So, it only remains the case $j = \lambda(n+1) + n$. Using the isomorphism

$$H^q(\mathbb{P}^{n+1}, i_*\Sigma(t)) \cong H^q(Q_n, \Sigma(t))$$

and Lemma 4.1; (i) – (ii), we obtain that $\text{Reg}^{CM}(i_*\Sigma(t)) = -t$. Therefore, if $j = \lambda(n+1) + n$, we have

$$\begin{aligned} \text{Reg}^{CM}(i_*\mathcal{E}_j) &= \text{Reg}^{CM}(i_*\Sigma(\lambda n + n - 1)) \\ &= -\lambda n - n + 1 \\ &= -j + \lambda + 1 \\ &= \text{Reg}_\sigma(\mathcal{E}_j) + \lambda + 1 \end{aligned}$$

where again the last equality follows from Proposition 3.5. \square

Proposition 4.5. *With the above notations, we have:*

$$\begin{aligned} (i) \quad &\text{Reg}^{CM}(i_*(\psi_1(3 + \lambda n))) = -2 - \lambda n. \\ (ii) \quad &\text{Reg}_\sigma(\psi_1(3 + \lambda n)) = -2 - \lambda(n + 1). \end{aligned}$$

Proof. (i) It follows from the fact that $H^i(Q_n, \psi_1(1 - i)) = 0$ for all $i \geq 1$ and

$$H^1(Q_n, \psi_1(-1)) \cong \mathbb{C}.$$

(ii) Since by Proposition 4.2 the right dual basis of $\sigma_{2+\lambda(n+1)-n}$ is

$$(\mathcal{O}_{Q_n}(\lambda n + 2), \psi_1^*(\lambda n + 2), \psi_2^*(\lambda n + 2), \Sigma(\lambda n + 2), \psi_{n-4}(\lambda n + 3), \dots, \psi_0(\lambda n + 3))$$

and the right dual basis of $\sigma_{3+\lambda(n+1)-n}$ is

$$(\mathcal{O}_{Q_n}(\lambda n + 3), \psi_1^*(\lambda n + 3), \psi_2^*(\lambda n + 3), \psi_3^*(\lambda n + 3), \Sigma(\lambda n + 3), \psi_{n-5}(\lambda n + 4), \dots, \psi_0(\lambda n + 4))$$

we easily check (taking into account that $\psi_0(\lambda n + 4) \cong \mathcal{O}_{Q_n}(\lambda n + 4)$ and that $H^1(Q_n, \psi_1(-1)) = \mathbb{C}$) that $\psi_1(3 + \lambda n)$ is $(-2 - \lambda(n + 1))$ -regular with respect to σ but not $(-3 - \lambda(n + 1))$ -regular with respect to σ . \square

5. FINAL COMMENTS

In this section, we will first gather the problems that arose from this paper and we will end with some final remarks.

Problem 5.1. To characterize smooth projective varieties which have a geometric collection.

Let X be a smooth projective variety of dimension n . We know that a necessary condition for having a geometric collection is that the rank of the Grothendieck group $K_0(X)$ is $n+1$ and X is Fano. We would like to know if these conditions are also sufficient.

The main goal of this paper was to generalize the notion of Castelnuovo-Mumford regularity for coherent sheaves on a projective space to coherent sheaves on other smooth projective varieties X . We have succeed provided X has a geometric collection of coherent sheaves. It would be nice to extend the definition to the case of smooth projective varieties which have a full strongly exceptional collection of sheaves (not necessarily geometric). Hence, we propose:

Problem 5.2. Let X be a smooth projective variety and let \mathcal{F} be a coherent sheaf on X . To extend the definition of regularity of \mathcal{F} with respect to a geometric collection σ to regularity of \mathcal{F} with respect to a full strongly exceptional collection.

It is well known that Beilinson's theorem ([1]) and Castelnuovo-Mumford regularity of sheaves play a fundamental role in the classification of vector bundles on projective spaces. In a forthcoming paper we will apply the results obtained in this work to study moduli spaces of vector bundles on quadric hypersurfaces. We have the feeling that Beilinson-Kapranov type spectral sequence (Theorem 2.21), Eilemberg-Moore type spectral sequence (Theorem 2.22) and the regularity with respect to a geometric collection will play an important role in the classification of vector bundles on varieties with a geometric collection.

REFERENCES

- [1] A.A. Beilinson, *Coherent sheaves on \mathbb{P}^n and Problems of Linear Algebra*, Funkt. Anal. Appl. **12** (1979), 214-216.
- [2] A.I. Bondal, *Representation of associative algebras and coherent sheaves*, Math. USSR Izvestiya **34** (1990), 23-42.
- [3] A.I. Bondal, A.E. Polishchuk *Homological properties of associative algebras: the method of helices*, Russian Acad. Sci. Izv. Math. **42** (1994), 219-259.
- [4] L. Costa and R.M. Miró-Roig, *Tilting sheaves on toric varieties*, Math. Z. **248** (2004), 849-865.
- [5] L. Costa and R.M. Miró-Roig, *Derived categories of projective bundles*, Proc. Amer. Math. Soc. **133** (2005), no. 9, 2533-2537.
- [6] L. Costa and R.M. Miró-Roig, *\mathbf{m} -blocks collections and Castelnuovo-Mumford regularity in multi-projective spaces*, Nagoya Math. J., to appear.
- [7] J.M. Drezet, J. Le Potier. *Fibrés stables et fibrés exceptionnelles sur \mathbb{P}^2* . Ann. Ec. Norm. Sup. **18** (1985), 193-244.
- [8] L. Gorodentsev, *Surgeries of exceptional bundles on \mathbb{P}^n* , Math. USSR Izv. **32** (1989), 1-13.
- [9] L. Gorodentsev, *Exceptional Vector Bundles on Surfaces with a moving Anticanonical Class*, Math. USSR Izv. **33** (1989), 67- 83.
- [10] A.L. Gorodentsev, S.A. Kuleshov, *Helix Theory*, Preprint MPI (2001), **97**.
- [11] L. Gorodentsev and A.N. Rudakov, *Exceptional Vector Bundles on the Projective Space*, Duke Math. J. **54**, (1987) 115-130.
- [12] R. Hartshorne, *Algebraic Geometry*, GTM **52**, Springer-Verlag (1983).
- [13] V.A. Iskovskih *Anticanonical models of three-dimensional varieties*, Journal of Soviet mathematics, 13 vol II (1980), 745-814.
- [14] M. M. Kapranov, *On the derived category of coherent sheaves on Grassmann manifolds*, Math. USSR Izvestiya, **24** (1985), 183-192.
- [15] M. M. Kapranov, *On the derived category of coherent sheaves on some homogeneous spaces*, Invent. Math., **92** (1988), 479-508.
- [16] A.G. Kuznetsov, *Exceptional collection of vector bundles on V_{22} Fano threefolds*, Moscow University, Mathematics Bulletin, **51** (1996), 35-37.
- [17] D. Mumford, *Lectures on curves on an algebraic surface*, Annals of Math. Studies **59** (1966) Princeton University Press.
- [18] D.Y. Nogin, *Spirals of Period four and Equations of Markov Type*, Math. USSR Izvestiya, **37** (1991), 209- 226.
- [19] D.Y. Nogin, *Helices on some Fano threefolds: Constructivity of semiorthogonal bases of K_0* , Ann. Sc. Ec. Norm. Sup. **27** (1994), 129-172.
- [20] D.O. Orlov, *Exceptional set of vector bundles on the variety V_5* . Vestnik Moskov. Univ. Ser. I Mat. Mekh. **5** (1991), 69-71.
- [21] G. Ottaviani, *Spinor bundles on quadrics*, Trans. AMS. **307**, (1988), 301-316.
- [22] A.N. Rudakov et al. *Helices and vector bundles: Seminaire Rudakov* Lecture Note Series, **148** (1990) Cambridge University Press.
- [23] A.N. Rudakov, *Exceptional vector Bundles on a Quadric*, Math. Izv. **33** (1989) 115-138.

FACULTAT DE MATEMÀTIQUES, DEPARTAMENT D'ÀLGEBRA I GEOMETRIA, GRAN VIA DE LES CORTS
CATALANES 585, 08007 BARCELONA, SPAIN

E-mail address: `costa@ub.edu`, `miro@ub.edu`